APPENDIX C

TABLE OF DUAL-INPUT DESCRIBING FUNCTIONS (DIDFs)

The DIDF sinusoidal gain is given by (cf. Sec. 6.1)

$$N_A(A,B,\omega) = n_p(A,B,\omega) + jn_q(A,B,\omega) = \frac{j}{\pi A}\int_0^{2\pi} y(B + A\sin\psi, A\omega\cos\psi)e^{-j\psi}\,d\psi$$

and the corresponding dc gain is given by (cf. Sec. 6.1)

$$N_B(A,B,\omega) = \frac{1}{2\pi B}\int_0^{2\pi} y(B + A\sin\psi, A\omega\cos\psi)\,d\psi$$

In this table we employ the "saturation function" (cf. Sec. 2.3) denoted by

$$\begin{aligned} f(\gamma) &= -1 && \gamma < -1 \\ &= \frac{2}{\pi}(\sin^{-1}\gamma + \gamma\sqrt{1-\gamma^2}) && |\gamma| \le 1 \\ &= 1 && \gamma > 1 \end{aligned}$$

and the associated function (cf. Sec. 6.2)

$$\begin{aligned} g(\gamma) &= \frac{2}{\pi}(\gamma\sin^{-1}\gamma + \sqrt{1-\gamma^2}) && |\gamma| \le 1 \\ &= |\gamma| && |\gamma| > 1 \end{aligned}$$

These functions are plotted in Fig. C.1. Two additional functions of considerable use are

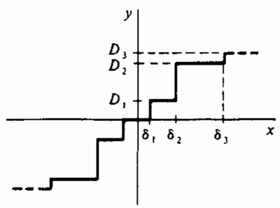
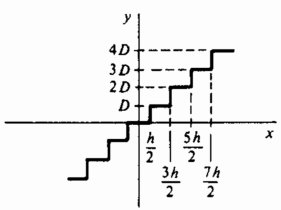
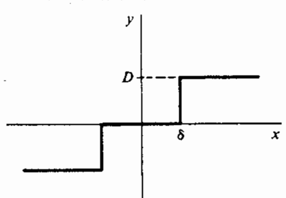
$$\begin{aligned} p(\gamma) &= -\tfrac{1}{2} && \gamma < -1 \\ &= \frac{1}{\pi}\sin^{-1}\gamma && |\gamma| \le 1 \\ &= \tfrac{1}{2} && \gamma > 1 \end{aligned}$$

and

$$\begin{aligned} q(\gamma) &= \frac{2}{\pi}\sqrt{1-\gamma^2} && |\gamma| \le 1 \\ &= 0 && |\gamma| > 1 \end{aligned}$$

These functions are plotted in Fig. C.2.

TABLE OF DUAL-INPUT DESCRIBING FUNCTIONS (DIDFs) *(Continued)*

Nonlinearity	Comments	$N_B(A,B,\omega)$, $n_p(A,B,\omega)$, and $n_q(A,B,\omega)$
 1. General odd quantizer	$\delta_{n+1} > A + \|B\| > \delta_n$ $D_0 = 0$	$N_B = \frac{1}{B}\sum_{i=1}^{n}(D_i - D_{i-1})\left[p\left(\frac{\delta_i + B}{A}\right) - p\left(\frac{\delta_i - B}{A}\right)\right]$ $n_p = \frac{1}{A}\sum_{i=1}^{n}(D_i - D_{i-1})\left[q\left(\frac{\delta_i + B}{A}\right) + q\left(\frac{\delta_i - B}{A}\right)\right]$ $n_q = 0$
 2. Uniform quantizer or granularity	$\frac{2n+1}{2}h > A + \|B\|$ $> \frac{2n-1}{2}h$	$N_B = \frac{D}{B}\sum_{i=1}^{n}\left[p\left(\frac{\frac{2i-1}{2}h + B}{A}\right) - p\left(\frac{\frac{2i-1}{2}h - B}{A}\right)\right]$ $n_p = \frac{D}{A}\sum_{i=1}^{n}\left[q\left(\frac{\frac{2i-1}{2}h + B}{A}\right) + q\left(\frac{\frac{2i-1}{2}h - B}{A}\right)\right]$ $n_q = 0$
 3. Relay with dead zone		$N_B = \frac{D}{B}\left[p\left(\frac{\delta + B}{A}\right) - p\left(\frac{\delta - B}{A}\right)\right]$ $n_p = \frac{D}{A}\left[q\left(\frac{\delta + B}{A}\right) + q\left(\frac{\delta - B}{A}\right)\right]$ $n_q = 0$

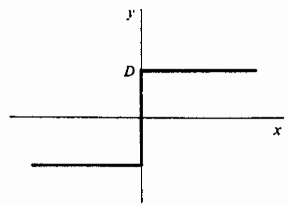
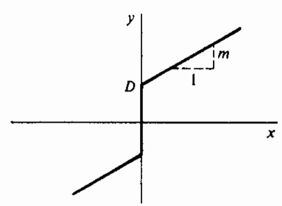
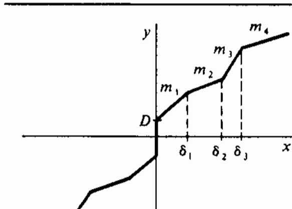
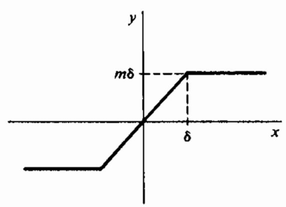
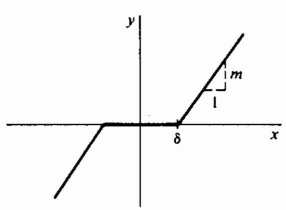
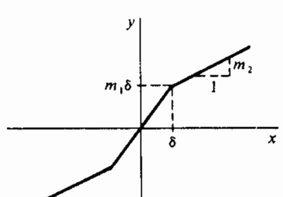
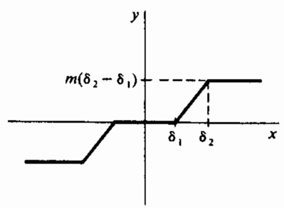
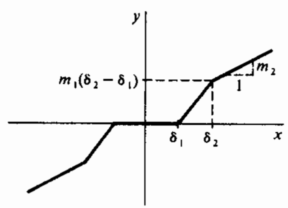
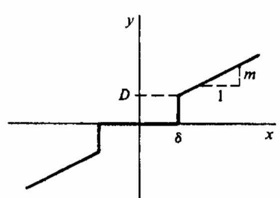
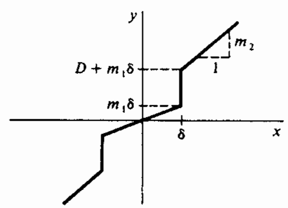
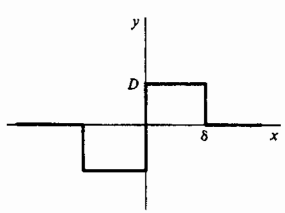
 4. Ideal relay	$A > \|B\|$ See Sec. 6.2	$N_B = \frac{2D}{\pi B} \sin^{-1} \frac{B}{A}$ $n_p = \frac{4D}{\pi A}\sqrt{1 - \left(\frac{B}{A}\right)^2}$ $n_q = 0$
 5. Preload	$A > \|B\|$	$N_B = \frac{2D}{\pi B} \sin^{-1} \frac{B}{A} + m$ $n_p = \frac{4D}{\pi A}\sqrt{1 - \left(\frac{B}{A}\right)^2} + m$ $n_q = 0$
 6. General piecewise-linear odd memoryless nonlinearity	$\delta_{n+1} > A + \|B\| > \delta_n$ and $A > \|B\|$	$N_B = \frac{2D}{\pi B} \sin^{-1} \frac{B}{A} + \frac{A}{2B} \sum_{i=1}^{n} (m_i - m_{i+1})\left[g\left(\frac{\delta_i + B}{A}\right) - g\left(\frac{\delta_i - B}{A}\right)\right] + m_{n+1}$ $n_p = \frac{4D}{\pi A}\sqrt{1 - \left(\frac{B}{A}\right)^2} + \frac{1}{2} \sum_{i=1}^{n} (m_i - m_{i+1})\left[f\left(\frac{\delta_i + B}{A}\right) + f\left(\frac{\delta_i - B}{A}\right)\right] + m_{n+1}$ $n_q = 0$

TABLE OF DUAL-INPUT DESCRIBING FUNCTIONS (DIDFs) *(Continued)*

Nonlinearity	Comments	$N_B(A,B,\omega)$, $n_p(A,B,\omega)$, and $n_q(A,B,\omega)$
 7. Saturation or limiter		$N_B = \frac{mA}{2B}\left[g\left(\frac{\delta+B}{A}\right) - g\left(\frac{\delta-B}{A}\right)\right]$ $n_p = \frac{m}{2}\left[f\left(\frac{\delta+B}{A}\right) + f\left(\frac{\delta-B}{A}\right)\right]$ $n_q = 0$
 8. Dead zone or threshold		$N_B = m\left\{1 - \frac{A}{2B}\left[g\left(\frac{\delta+B}{A}\right) - g\left(\frac{\delta-B}{A}\right)\right]\right\}$ $n_p = m\left\{1 - \frac{1}{2}\left[f\left(\frac{\delta+B}{A}\right) + f\left(\frac{\delta-B}{A}\right)\right]\right\}$ $n_q = 0$
 9. Gain-changing nonlinearity		$N_B = \frac{A}{2B}(m_1 - m_2)\left[g\left(\frac{\delta+B}{A}\right) - g\left(\frac{\delta-B}{A}\right)\right] + m_2$ $n_p = \frac{1}{2}(m_1 - m_2)\left[f\left(\frac{\delta+B}{A}\right) + f\left(\frac{\delta-B}{A}\right)\right] + m_2$ $n_q = 0$

 10. Limiter with dead zone		$N_B = \frac{mA}{2B}\left[g\left(\frac{\delta_2+B}{A}\right) - g\left(\frac{\delta_2-B}{A}\right) - g\left(\frac{\delta_1+B}{A}\right) + g\left(\frac{\delta_1-B}{A}\right)\right]$ $n_p = \frac{m}{2}\left[f\left(\frac{\delta_2+B}{A}\right) + f\left(\frac{\delta_2-B}{A}\right) - f\left(\frac{\delta_1+B}{A}\right) - f\left(\frac{\delta_1-B}{A}\right)\right]$ $n_q = 0$
 11. Gain-changing nonlinearity with dead zone		$N_B = \frac{A}{2B}\left\{(m_1-m_2)\left[g\left(\frac{\delta_2+B}{A}\right) - g\left(\frac{\delta_2-B}{A}\right)\right] - m_1\left[g\left(\frac{\delta_1+B}{A}\right) - g\left(\frac{\delta_1-B}{A}\right)\right]\right\} + m_2$ $n_p = \frac{m_1-m_2}{2}\left[f\left(\frac{\delta_2+B}{A}\right) + f\left(\frac{\delta_2-B}{A}\right)\right] - \frac{m_1}{2}\left[f\left(\frac{\delta_1+B}{A}\right) + f\left(\frac{\delta_1-B}{A}\right)\right] + m_2$ $n_q = 0$
 12.		$N_B = m\left\{1 - \frac{A}{2B}\left[g\left(\frac{\delta+B}{A}\right) - g\left(\frac{\delta-B}{A}\right)\right]\right\} + \frac{D}{B}\left[p\left(\frac{\delta+B}{A}\right) - p\left(\frac{\delta-B}{A}\right)\right]$ $n_p = m\left\{1 - \frac{1}{2}\left[f\left(\frac{\delta+B}{A}\right) + f\left(\frac{\delta-B}{A}\right)\right]\right\} + \frac{D}{A}\left[q\left(\frac{\delta+B}{A}\right) + q\left(\frac{\delta-B}{A}\right)\right]$ $n_q = 0$

Nonlinearity	Comments	$N_B(A,B,\omega)$, $n_p(A,B,\omega)$, and $n_q(A,B,\omega)$
13. 		$N_B = \frac{A}{2B}(m_1 - m_2)\left[g\left(\frac{\delta + B}{A}\right) - g\left(\frac{\delta - B}{A}\right)\right] + m_2 + \frac{D}{B}\left[p\left(\frac{\delta + B}{A}\right) - p\left(\frac{\delta - B}{A}\right)\right]$ $n_p = \frac{1}{2}(m_1 - m_2)\left[f\left(\frac{\delta + B}{A}\right) + f\left(\frac{\delta - B}{A}\right)\right] + m_2 + \frac{D}{A}\left[q\left(\frac{\delta + B}{A}\right) + q\left(\frac{\delta - B}{A}\right)\right]$ $n_q = 0$
14. 	$A > \|B\|$	$N_B = \frac{D}{B}\left[\frac{2}{\pi}\sin^{-1}\frac{B}{A} - p\left(\frac{\delta + B}{A}\right) + p\left(\frac{\delta - B}{A}\right)\right]$ $n_p = \frac{D}{A}\left[\frac{4}{\pi}\sqrt{1 - \left(\frac{B}{A}\right)^2} - q\left(\frac{\delta + B}{A}\right) - q\left(\frac{\delta - B}{A}\right)\right]$ $n_q = 0$

$y = c$ 15.		$N_B = \dfrac{c}{B}$ $n_p = 0$ $n_q = 0$
$y = x$ 16. Linear gain		$N_B = 1$ $n_p = 1$ $n_q = 0$
$y = x\lvert x\rvert$ 17. Odd square law		$N_B = \dfrac{A^2}{\pi B}\left\{\left[1 + 2\left(\dfrac{B}{A}\right)^2\right]\sin^{-1}\dfrac{B}{A} + 3\dfrac{B}{A}\sqrt{1 - \left(\dfrac{B}{A}\right)^2}\right\}$ $n_p = \dfrac{8}{3\pi}A\left\{\left[1 + \dfrac{1}{2}\left(\dfrac{B}{A}\right)^2\right]\sqrt{1 - \left(\dfrac{B}{A}\right)^2} + \dfrac{3}{2}\dfrac{B}{A}\sin^{-1}\dfrac{B}{A}\right\}$ $n_q = 0$
$y = x^3$ 18. Cubic characteristic	See Sec. 6.2	$N_B = \frac{3}{2}A^2 + B^2$ $n_p = \frac{3}{4}A^2 + 3B^2$ $n_q = 0$
$y = x^5$ 20. Quintic characteristic		$N_B = \frac{15}{8}A^4 + 5A^2B^2 + B^4$ $n_p = \frac{5}{8}A^4 + \frac{15}{2}A^2B^2 + 5B^4$ $n_q = 0$
$y = x^7$ 22.		$N_B = \frac{35}{16}A^6 + \frac{105}{8}A^4B^2 + \frac{21}{2}A^2B^4 + B^6$ $n_p = \frac{35}{64}A^6 + \frac{105}{8}A^4B^2 + \frac{105}{4}A^2B^4 + 7B^6$ $n_q = 0$

TABLE OF DUAL-INPUT DESCRIBING FUNCTIONS (DIDFs) *(Continued)*

Nonlinearity	Comments	$N_B(A,B,\omega)$, $n_p(A,B,\omega)$, and $n_q(A,B,\omega)$
$y = x^n$ 24.	$n = 3, 5, 7, \ldots$ See Sec. 6.2	$N_B = \frac{1}{\sqrt{\pi}} \sum_{k(\mathrm{odd})=1}^{n} \frac{n!}{(n-k)!\,k!} A^{n-k}B^{k-1} \frac{\Gamma\left(\frac{n-k+1}{2}\right)}{\Gamma\left(\frac{n-k+2}{2}\right)}$ $n_p = \frac{2}{\sqrt{\pi}} \sum_{k(\mathrm{even})=0}^{n-1} \frac{n!}{(n-k)!\,k!} A^{n-k-1}B^{k} \frac{\Gamma\left(\frac{n-k+2}{2}\right)}{\Gamma\left(\frac{n-k+3}{2}\right)}$ $n_q = 0$
$y = M \sin mx$ 29. Harmonic nonlinearity		$N_B = \frac{M}{B} J_0(mA) \sin mB$ $n_p = \frac{2M}{A} J_1(mA) \cos mB$ $n_q = 0$
$y = M \sinh mx$ 30.	I_0 and I_1 are modified Bessel functions of orders 0 and 1, respectively.	$N_B = \frac{M}{B} I_0(mA) \sinh mB$ $n_p = \frac{2M}{A} I_1(mA) \cosh mB$ $n_q = 0$

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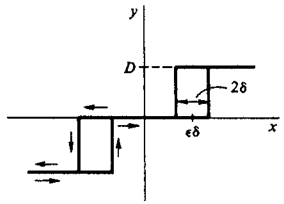
<td>

37. (positive) Hysteresis</td>
<td>$A - |B| > \delta(1 + \epsilon)$</td>
<td>$N_B = 0$
$n_q = 0$
$n_q = 0$
$$N_B = \frac{D}{2\pi B}\left[\sin^{-1}\frac{(\epsilon + 1)\delta + B}{A} - \sin^{-1}\frac{(\epsilon + 1)\delta - B}{A} + \sin^{-1}\frac{(\epsilon - 1)\delta + B}{A} - \sin^{-1}\frac{(\epsilon - 1)\delta - B}{A}\right]$$
$$n_p = \frac{D}{\pi A}\left\{\sqrt{1 - \left[\frac{(\epsilon + 1)\delta + B}{A}\right]^2} + \sqrt{1 - \left[\frac{(\epsilon + 1)\delta - B}{A}\right]^2} + \sqrt{1 - \left[\frac{(\epsilon - 1)\delta + B}{A}\right]^2} + \sqrt{1 - \left[\frac{(\epsilon - 1)\delta - B}{A}\right]^2}\right\}$$
$$n_q = -\frac{4D\delta}{\pi A^2}$$</td>
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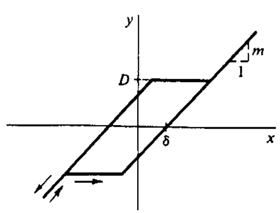
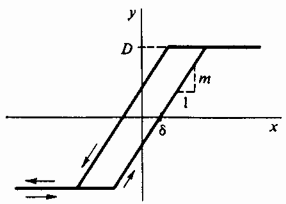
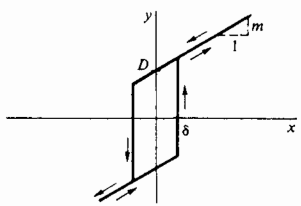
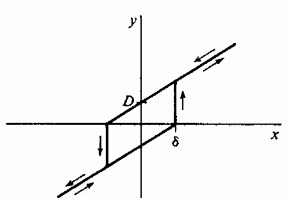
<td>

40.</td>
<td>$A - |B| > \frac{D}{m} + \delta$</td>
<td>$$N_B = m\left\{1 + \frac{A}{4B}\left[-g\left(\frac{\frac{D}{m} + \delta + B}{A}\right) + g\left(\frac{\frac{D}{m} + \delta - B}{A}\right) + g\left(\frac{\frac{D}{m} - \delta + B}{A}\right) - g\left(\frac{\frac{D}{m} - \delta - B}{A}\right)\right]\right\}$$
$$n_p = m\left\{1 - \frac{1}{4}\left[f\left(\frac{\frac{D}{m} + \delta + B}{A}\right) + f\left(\frac{\frac{D}{m} + \delta - B}{A}\right) - f\left(\frac{\frac{D}{m} - \delta + B}{A}\right) - f\left(\frac{\frac{D}{m} - \delta - B}{A}\right)\right]\right\}$$
$$n_q = -\frac{4D\delta}{\pi A^2}$$</td>
</tr>
</table>

Nonlinearity	Comments	$N_B(A,B,\omega)$, $n_p(A,B,\omega)$, and $n_q(A,B,\omega)$
 42.	$A - \lvert B \rvert > \dfrac{D}{m} + \delta$	$N_B = \dfrac{m}{4}\left[g\left(\dfrac{\frac{D}{m}+\delta+B}{A}\right) - g\left(\dfrac{\frac{D}{m}+\delta-B}{A}\right) + g\left(\dfrac{\frac{D}{m}-\delta+B}{A}\right) - g\left(\dfrac{\frac{D}{m}-\delta-B}{A}\right)\right]$ $n_p = \dfrac{m}{4}\left[f\left(\dfrac{\frac{D}{m}+\delta+B}{A}\right) + f\left(\dfrac{\frac{D}{m}+\delta-B}{A}\right) + f\left(\dfrac{\frac{D}{m}-\delta+B}{A}\right) + f\left(\dfrac{\frac{D}{m}-\delta-B}{A}\right)\right]$ $n_q = -\dfrac{4D\delta}{\pi A^2}$
 43.	$A - \lvert B \rvert > \delta$	$N_B = \dfrac{D}{\pi B}\left(\sin^{-1}\dfrac{\delta+B}{A} - \sin^{-1}\dfrac{\delta-B}{A}\right) + m$ $n_p = \dfrac{2D}{\pi A}\left[\sqrt{1-\left(\dfrac{\delta+B}{A}\right)^2} + \sqrt{1-\left(\dfrac{\delta-B}{A}\right)^2}\right] + m$ $n_q = -\dfrac{4D\delta}{\pi A^2}$
 44. Negative deficiency	$A - \lvert B \rvert > \delta$	$N_B = \dfrac{D}{\pi B}\left(\sin^{-1}\dfrac{\delta+B}{A} - \sin^{-1}\dfrac{\delta-B}{A}\right) + \dfrac{D}{\delta}$ $n_p = \dfrac{2D}{\pi A}\left[\sqrt{1-\left(\dfrac{\delta+B}{A}\right)^2} + \sqrt{1-\left(\dfrac{\delta-B}{A}\right)^2}\right] + \dfrac{D}{\delta}$ $n_q = -\dfrac{4D\delta}{\pi A^2}$

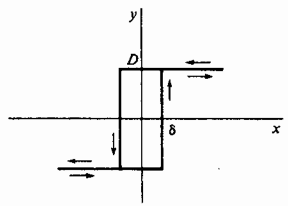
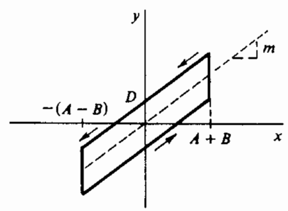
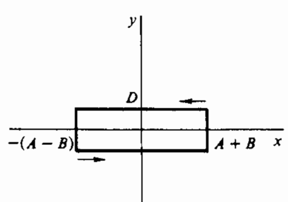
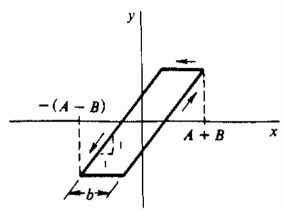
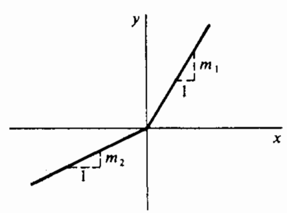
 45. Rectangular hysteresis or toggle	$A - \|B\| > \delta$ See Sec. 6.2	$N_B = \frac{D}{\pi B}\left(\sin^{-1}\frac{\delta + B}{A} - \sin^{-1}\frac{\delta - B}{A}\right)$ $n_p = \frac{2D}{\pi A}\left[\sqrt{1 - \left(\frac{\delta + B}{A}\right)^2} + \sqrt{1 - \left(\frac{\delta - B}{A}\right)^2}\right]$ $n_q = -\frac{4D\delta}{\pi A^2}$
 46.		$N_B = m$ $n_p = m$ $n_q = -\frac{4D}{\pi A}$
 47.		$N_B = 0$ $n_p = 0$ $n_q = -\frac{4D}{\pi A}$

TABLE OF DUAL-INPUT DESCRIBING FUNCTIONS (DIDFs) *(Continued)*

Nonlinearity	Comments	$N_B(A,B,\omega)$, $n_p(A,B,\omega)$, and $n_q(A,B,\omega)$
 48. Friction-controlled backlash		$N_B = 1$ $n_p = \frac{1}{2}\left[1 + f\left(1 - \frac{b}{A}\right)\right]$ $n_q = -\frac{1}{\pi}\left[2\frac{b}{A} - \left(\frac{b}{A}\right)^2\right]$
 51.	$B > A$ $-A < B < A$ $B < -A$ See Sec. 6.2	$N_B = m_1$ $n_p = m_1$ $n_q = 0$ $N_B = \frac{m_1 + m_2}{2} + \frac{m_1 - m_2}{2}\frac{A}{B}g\left(\frac{B}{A}\right)$ $n_p = \frac{m_1 + m_2}{2} + \frac{m_1 - m_2}{2}f\left(\frac{B}{A}\right)$ $n_q = 0$ $N_B = m_2$ $n_p = m_2$ $n_q = 0$

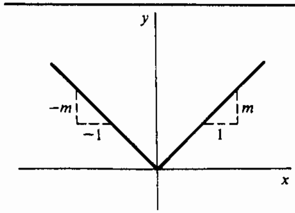
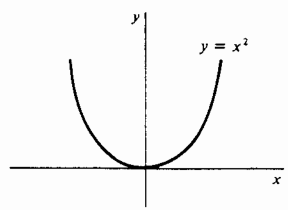
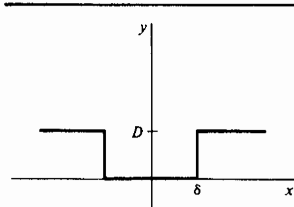
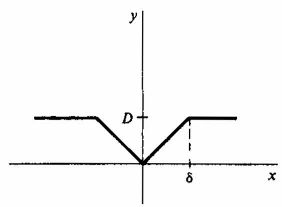
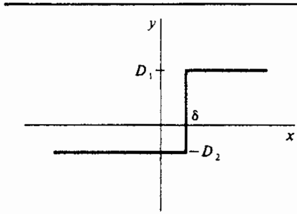
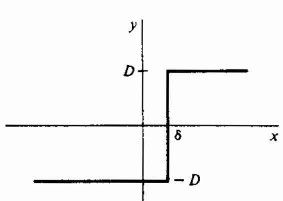
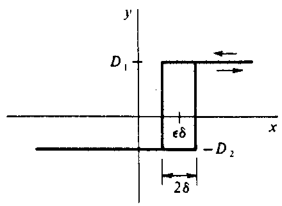
 52. Absolute value	$B > A$	$N_B = m$ $n_p = m$ $n_q = 0$
	$-A < B < A$	$N_B = m\dfrac{A}{B}g\left(\dfrac{B}{A}\right)$ $n_p = mf\left(\dfrac{B}{A}\right)$ $n_q = 0$
	$B < -A$	$N_B = -m$ $n_p = -m$ $n_q = 0$
 53. Square law		$N_B = \dfrac{1}{B}[B^2 + \tfrac{1}{2}A^2]$ $n_p = 2B$ $n_q = 0$
 54.	$A + \lvert B\rvert < \delta$	$N_B = 0$ $n_p = 0$ $n_q = 0$
	$A - \lvert B\rvert > \delta$	$N_B = \dfrac{D}{\pi B}\left(\pi - \sin^{-1}\dfrac{\delta - B}{A} - \sin^{-1}\dfrac{\delta + B}{A}\right)$ $n_p = \dfrac{2D}{\pi A}\left[\sqrt{1 - \left(\dfrac{\delta - B}{A}\right)^2} - \sqrt{1 - \left(\dfrac{\delta + B}{A}\right)^2}\right]$ $n_q = 0$

TABLE OF DUAL-INPUT DESCRIBING FUNCTIONS (DIDFs) *(Continued)*

Nonlinearity	Comments	$N_B(A,B,\omega)$, $n_p(A,B,\omega)$, and $n_q(A,B,\omega)$
 55.	$A + \|B\| < \delta$ $A - \|B\| > \delta$	N_B (see absolute-value case) n_p (see absolute-value case) $n_q = 0$ $N_B = \frac{D}{2\delta}\frac{A}{B}\left[-g\left(\frac{\delta - B}{A}\right) + 2g\left(\frac{B}{A}\right) - g\left(\frac{\delta + B}{A}\right)\right] + \frac{D}{B}$ $n_p = \frac{D}{\pi\delta}\left[f\left(\frac{\delta - B}{A}\right) + 2f\left(\frac{B}{A}\right) - f\left(\frac{\delta + B}{A}\right)\right]$ $\eta_q = 0$
 56. Input- and output-biased ideal relay	$A - \|B\| > \delta$	$N_B = \frac{D_1 - D_2}{2B} - \frac{D_1 + D_2}{\pi B}\sin^{-1}\frac{\delta - B}{A}$ $n_p = \frac{2(D_1 + D_2)}{\pi A}\sqrt{1 - \left(\frac{\delta - B}{A}\right)^2}$ $n_q = 0$
 57. Input-biased ideal relay	$A - \|B\| > \delta$	$N_B = \frac{2D}{\pi B}\sin^{-1}\frac{B - \delta}{A}$ $n_p = \frac{4D}{\pi A}\sqrt{1 - \left(\frac{\delta - B}{A}\right)^2}$ $n_q = 0$

 58. Input- and output-biased rectangular hysteresis	$A - \lvert B \rvert > \delta(\epsilon + 1)$	$N_B = \dfrac{1}{2\pi B}\left\{(D_1 + D_2)\left[\pi - \sin^{-1}\dfrac{\delta(\epsilon - 1) - B}{A} - \sin^{-1}\dfrac{\delta(\epsilon + 1) - B}{A}\right] - 2\pi D_2\right\}$ $n_p = \dfrac{D_1 + D_2}{\pi A}\left\{\sqrt{1 - \left[\dfrac{\delta(\epsilon + 1) - B}{A}\right]^2} + \sqrt{1 - \left[\dfrac{\delta(\epsilon - 1) - B}{A}\right]^2}\right\}$ $n_q = -\dfrac{2(D_1 + D_2)\delta}{\pi A^2}$

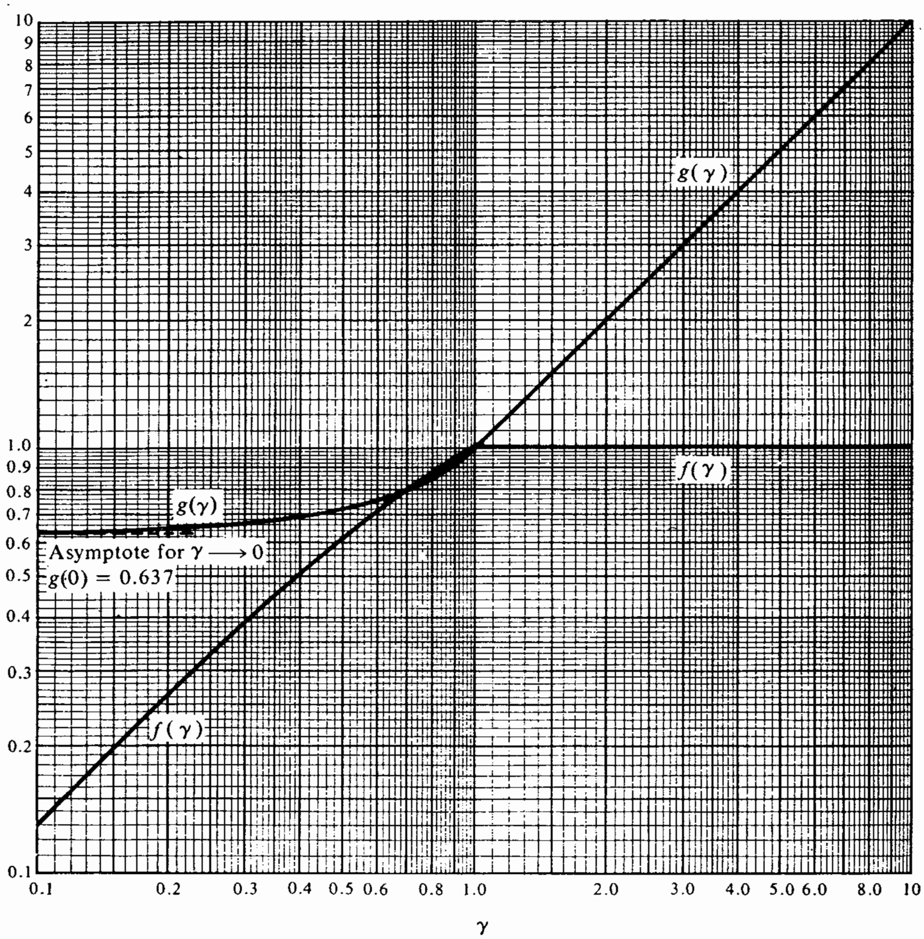


Figure C.1 *Graphs of $f(\gamma)$ and $g(\gamma)$.*

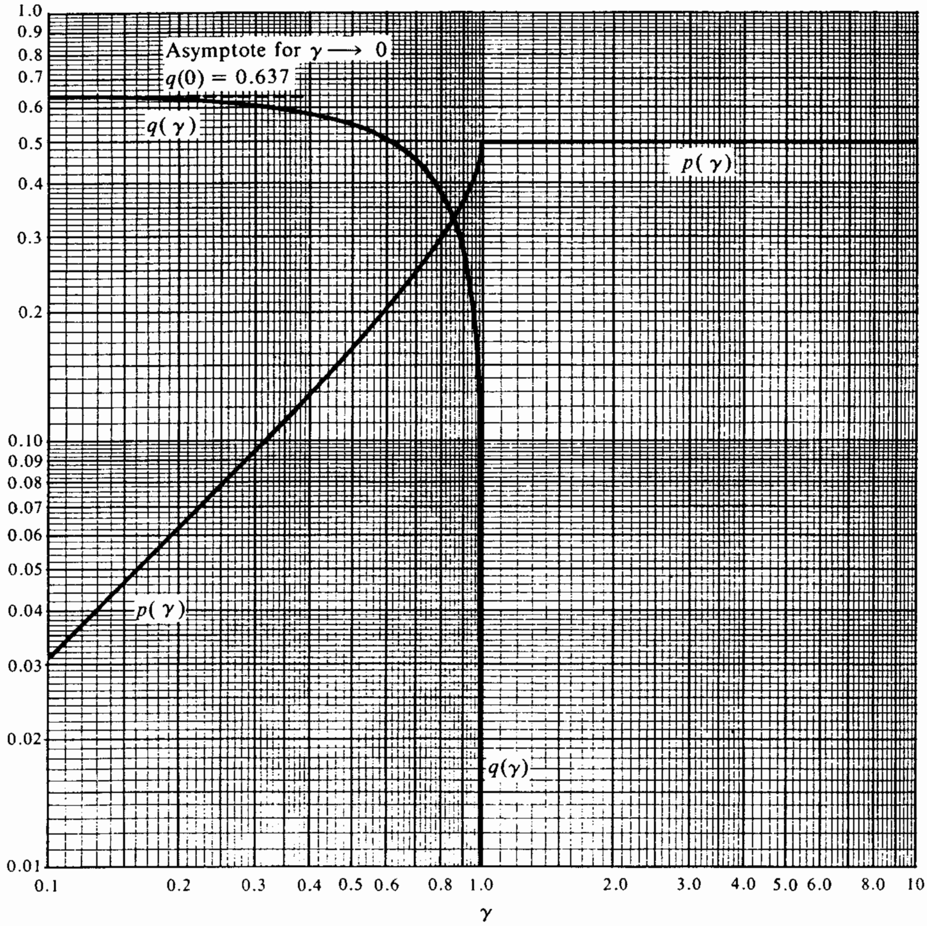


Figure C.2 *Graphs of $p(\gamma)$ and $q(\gamma)$.*