Lecture 4

Last time: Left off with characteristic function.

4. Prove $\phi_x(t) = \Pi \phi_{x_i}(t)$ where $X = X_1 + X_2 + ... + X_n$ (X_i independent)

Let $S = X_1 + X_2 + ... X_n$ where the X_i are <u>independent</u>.

$$\phi_s(t) = E\left[e^{jtS}\right] = E\left[e^{jt(X_1+X_2+...+X_n)}\right]$$
$$= E\left[e^{jtX_1}\right]E\left[e^{jtX_2}\right]...E\left[e^{jtX_n}\right]$$
$$= \prod \phi_{X_i}(t)$$

This is the main reason why use of the characteristic function is convenient. This would also follow from the more devious reasoning of the density function for the sum of n independent random variables being the n^{th} order convolution of the individual density functions – and the knowledge that convolution in the direct variable domain becomes multiplication in the transform domain.

5. MacLaurin series expansion of $\phi(t)$

Because $f(x)$ is non-negative and $\int_{-\infty}^{\infty} f(x)dx = 1$ (or, even better, $\int_{-\infty}^{\infty} |f(x)|dx = 1$), it follows that $\int_{-\infty}^{\infty} |f(x)|dx = 1$ converges so that $f(x)$ is Fourier transformable. Thus the characteristic function $\phi(t)$ exists for all distributions and the inverse relation $\phi(t) \rightarrow f(x)$ holds for all distributions. This implies that $\phi(t)$ is analytic for all real values of t.

Then it can be expanded in a power series, which converges for all finite values of t.

$$\phi(t) = \phi(0) + \phi^{(1)}(0)t + \frac{1}{2!}\phi^{(2)}(0)t^2 + ... + \frac{1}{n!}\phi^{(n)}(0)t^n + ...$$
$$\phi(t) = \int_{-\infty}^{\infty} f(x)e^{jtx}dx\,, \quad \phi(0) = 1$$

$$\frac{d^n\phi(t)}{dt^n} = \int_{-\infty}^{\infty} f(x)(jx)^n e^{jtx}dx$$

$$\phi^{(n)}(0) = j^n \int_{-\infty}^{\infty} x^n f(x)dx = j^n \overline{X^n}$$

$$\phi(t) = 1 + j\overline{X}t + \frac{1}{2!}\left(j^2\right)\overline{X^2}t^2 + \ldots + \frac{1}{n!}\left(j^n\right)\overline{X^n}t^n + \ldots$$

The coefficients of the expansion are given by the moments of the distribution. Thus the characteristic function can be determined from the moments. Similarly, the moments can be determined from the characteristic function directly by

$$\overline{X^n} = \frac{1}{j^n}\frac{d^n\phi(t)}{dt^n}\bigg|_{t=0}$$

or by expanding $\phi(t)$ into its power series in some other way and identifying the coefficients of the various powers of t.

The Generating Function

The generating function has its most useful application to random variables which take integer values only. Examples of such would be the number of telephone calls into a switchboard in a certain time interval, the number of cars entering a toll station in a certain time interval, the number of times a 7 is thrown in n tosses of 2 dice, etc.

For integer-valued random variables, the Generating Function yields the same advantages as the Characteristic Function and is of simpler form.

Consider a random variable which takes the integer values k:

$P(X = k) = p_k$ (k=0,1,2,…)

For a discrete distribution you can sum in lieu of integration. The Characteristic Function for this random variable is

$$\phi(t) = E\left[e^{jtX}\right] = \sum_{k=0}^{\infty} e^{jtk} p_k$$

$$= \sum_{k=0}^{\infty} p_k \left(e^{jt}\right)^k$$

If we define a new variable $s = e^{jt}$, we have

$$G(s) = \sum_{k=0}^{\infty} p_k s^k$$

which is called the Generating Function. It has all the interesting properties of the characteristic function. Note that $t \to 0$ corresponds to $s \to 1$.

Let's establish the connection between moments of a distribution and the generating function:

$$\frac{dG}{ds} = \sum_{k=0}^{\infty} k p_k s^{k-1}$$

$$\frac{d^2G}{ds^2} = \sum_{k=0}^{\infty} k(k-1) p_k s^{k-2}$$

$$= \sum_{k=0}^{\infty} k^2 p_k s^{k-2} - \sum_{k=0}^{\infty} k p_k s^{k-2}$$

Just calculate $\left.\frac{dG}{ds}\right|_{s=1}$ and $\left.\frac{d^2G}{ds^2}\right|_{s=1}$ and reorganize them in terms of $\bar{X}$ and $\overline{X^2}$:

$$\left.\frac{dG}{ds}\right|_{s=1} = \sum_{k=0}^{\infty} k p_k = \bar{X}, \;\leftarrow\; 1^{\text{st}} \text{ moment expression}$$

$$\left.\frac{d^2G}{ds^2}\right|_{s=1} = \sum_{k=0}^{\infty} k^2 p_k - \sum_{k=0}^{\infty} k p_k$$

$$\overline{X^2} = \left.\frac{d^2G}{ds^2}\right|_{s=1} + \left.\frac{dG}{ds}\right|_{s=1} \leftarrow 2^{\text{nd}} \text{ moment expression}$$

Each moment is a linear combination of its order derivative and lower order derivatives. The generating function for the sum of independent integer-valued variables is the product of their generating functions. This is harder to prove than the same property of the characteristic function, but it does, in fact, hold true.

Multiple Random Variables

Characterizing a joint set of random variables, define a probability distribution function

$$F(\underline{x}) = P(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n)$$

This is called the *joint probability distribution function*.

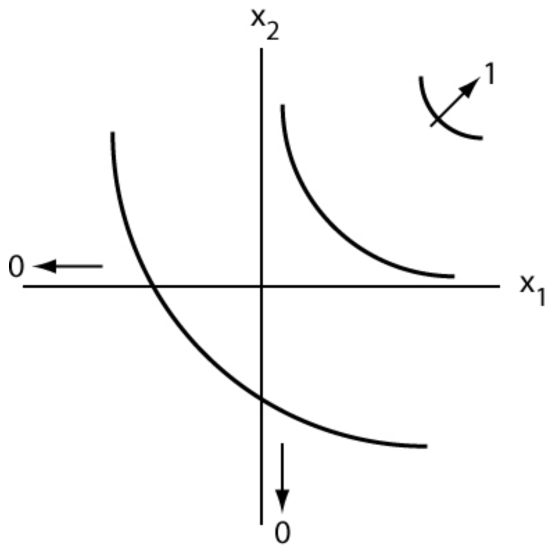
Properties:

If any of the arguments x_i goes to $-\infty$, then $F(\underline{x}) \to 0$.

$$\lim_{\text{any } x_i \to -\infty} F(\underline{x}) = 0$$

If all of the x_i go to ∞, then $F(\underline{x}) \rightarrow 1$.

$\lim_{\text{all } x_i \rightarrow \infty} F(\underline{x}) = 1$



$F(\underline{x})$ is monotonically non-decreasing in each x_i.

Define joint density function by differentiation:

$$f(\underline{x}) = \frac{\partial^n}{\partial x_1 \partial x_2 \ldots \partial x_n}$$

$$f(\underline{x}) \geq 0, \forall \underline{x}$$

$$F_{x_1 \ldots x_n}(x_1 \ldots x_n) = \int_{-\infty}^{x_1} du_1 \ldots \int_{-\infty}^{x_n} du_n f_{x_1 \ldots x_n}(u_1 \ldots u_n)$$

Setting each $x_i \rightarrow \infty$,

$$\int_{-\infty}^{\infty} du_1 \ldots \int_{-\infty}^{\infty} du_n F_{u_1, \ldots, u_n}(u_1, \ldots, u_n) = 1$$

$$\begin{aligned} F_{x_1, \ldots, x_k}(x_1, \ldots, x_k) &= P(X_1 \leq x_1, \ldots, X_n \leq x_n) \\ &= P(X_1 \leq x_1, \ldots, X_k \leq x_k, X_{k+1} \leq \infty, \ldots, X_n \leq \infty) \\ &= F_{x_1, \ldots, x_n}(x_1, \ldots, x_k, \infty, \ldots, \infty) \end{aligned}$$

For the density function:

$$\begin{aligned}
f_{x_1,\dots,x_k}(x_1,\dots,x_k) &= \frac{\partial^k}{\partial x_1 \partial x_2 \dots \partial x_k} F_{x_1,\dots,x_k}(x_1,\dots,x_k) \\
&= \frac{\partial^k}{\partial x_1 \partial x_2 \dots \partial x_k} F_{x_1,\dots,x_n}(x_1,\dots,x_k,\infty,\dots,\infty) \\
&= \frac{\partial^k}{\partial x_1 \partial x_2 \dots \partial x_k} \int_{-\infty}^{x_1} du_1 \dots \int_{-\infty}^{x_k} du_k \int_{-\infty}^{\infty} du_{k+1} \dots \int_{-\infty}^{\infty} du_n f_{x_1,\dots,x_n}(u_1,\dots,u_n) \\
&= \int_{-\infty}^{\infty} du_{k+1} \dots \int_{-\infty}^{\infty} du_n f_{x_1,\dots,x_n}(x_1,\dots,x_k,u_{k+1},\dots,u_n) \\
&= \int_{-\infty}^{\infty} du_{k+1} \dots \int_{-\infty}^{\infty} du_n f_{x_1,\dots,x_n}(x_1,\dots,x_n)
\end{aligned}$$

Marginal density

If you integrate above over all variables but one, it is referred to as the *marginal density*.

$$f_{x_i}(x_i) = \underbrace{\int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_n}_{\text{n-1 terms: all except } x_i} \; f_{x_1,\dots,x_n}(x_1,\dots,x_n)$$

Mutually independent sets of random variables

Definition of independence:

$$P\left[X_1 \in s_1, X_2 \in s_2, \dots\right] = P\left[X_1 \in s_1\right] P\left[X_2 \in s_2\right] \dots$$

for any sets s_1, s_2, …

The <u>product rule</u> holds for joint probability distribution and density functions for independent random variables.

$$F_{x_1,x_2,x_3,\dots}(x_1,x_2,x_3,\dots) = F_{x_1}(x_1) F_{x_2}(x_2) F_{x_2}(x_2) \dots$$

$$f_{x_1,x_2,x_3,\dots}(x_1,x_2,x_3,\dots) = f_{x_1}(x_1) f_{x_2}(x_2) f_{x_2}(x_2) \dots$$

<u>Expectations</u>

$$E\left[g(\underline{x})\right] = \int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_n g(\underline{x}) f(\underline{x})$$

For the sum of multiple random variables:

$$\begin{aligned}
E\left[X_1+X_2+\ldots+X_n\right] &= \int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_n (x_1+x_2+\ldots+x_n) f_{x_1,\ldots,x_n}(x_1,\ldots,x_n) \\
&= \int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_n x_1 f_{x_1,\ldots,x_n}(x_1,\ldots,x_n) + \ldots + \int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_n x_n f_{x_1,\ldots,x_n}(x_1,\ldots,x_n) \\
&= \int_{-\infty}^{\infty} x_1 f_{x_1}(x_1) dx_1 + \int_{-\infty}^{\infty} x_2 f_{x_2}(x_2) dx_2 + \ldots + \int_{-\infty}^{\infty} x_n f_{x_n}(x_n) dx_n \\
&= E\left[X_1\right] + E\left[X_2\right] + \ldots + E\left[X_n\right]
\end{aligned}$$

This relation is true whether or not the x_i are independent.

For the product of multiple <u>independent</u> random variables:

$$\begin{aligned}
E\left[X_1 X_2 \ldots X_n\right] &= \int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_n (x_1 x_2 \ldots x_n) f_{x_1,\ldots,x_n}(x_1,\ldots,x_n) \\
&= \int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_n (x_1 x_2 \ldots x_n) f_{x_1}(x_1) f_{x_2}(x_2) \ldots f_{x_n}(x_n) \\
&= \int_{-\infty}^{\infty} x_1 f_{x_1}(x_1) dx_1 \int_{-\infty}^{\infty} x_2 f_{x_2}(x_2) dx_2 \ldots \int_{-\infty}^{\infty} x_n f_{x_n}(x_n) dx_n \\
&= E\left[X_1\right] E\left[X_2\right] \ldots E\left[X_n\right]
\end{aligned}$$