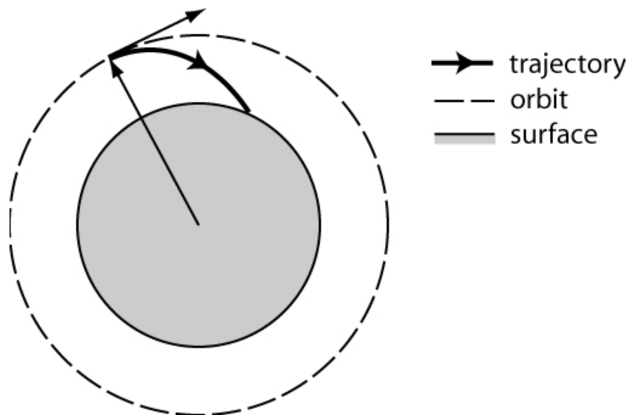
Lecture 9

Last time: Linearized error propagation



$\underline{e}_s = S\underline{e}_1$

Integrate the errors at deployment to find the error at the surface.

$$
\begin{aligned}
E_s &= \overline{\underline{e}_s \underline{e}_s^T} \\
&= S\overline{\underline{e}_1 \underline{e}_1^T} S^T \\
&= SE_1 S^T
\end{aligned}
$$

Or Φ can be integrated from:

$$\dot{\Phi} = F\Phi, \quad \text{where } \Phi(0) = I$$

$$\underline{\dot{x}} = \underline{f}(\underline{x})$$

$$F = \frac{d\underline{f}}{d\underline{x}}$$

where F is the linearized system matrix. But this requires the full Φ (same number of equations as finite differencing).

$t_n =$ time when the nominal trajectory impacts.

$\underline{e}(t_n) = \Phi(t_n)\underline{e}_1$

$\underline{e}_r(t_n) = \underline{e}_2 = \Phi_r \underline{e}_1$

where Φ_r is the upper 3 rows of $\Phi(t_n)$.

Covariance matrix:

$E_2 = \Phi_r E_1 \Phi_r^T$

$$\dot{\underline{e}} = F\underline{e}$$

$$E(t) = \overline{\underline{e}(t)\underline{e}(t)^T}$$

$$\dot{E}(t) = \overline{\dot{\underline{e}}(t)\underline{e}(t)^T} + \overline{\underline{e}(t)\dot{\underline{e}}(t)^T}$$

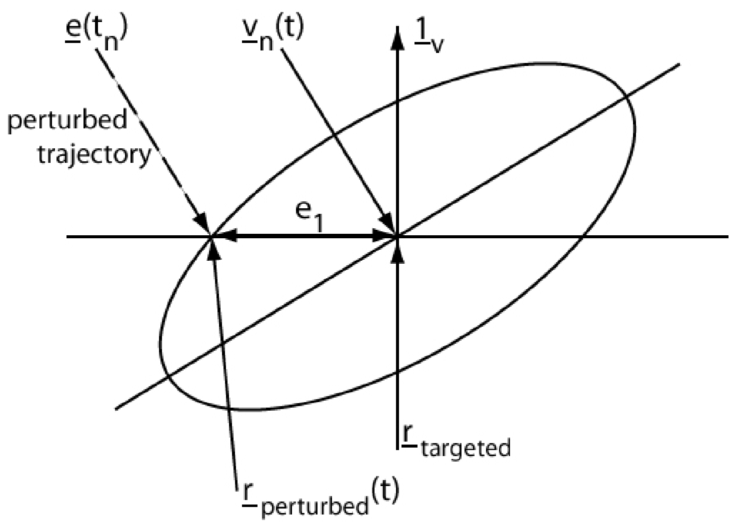
$$= F\overline{\underline{e}(t)\underline{e}(t)^T} + \overline{\underline{e}(t)\underline{e}(t)^T}F^T$$

$$= FE(t) + E(t)F^T$$

You can integrate this differential equation to t_n from $E(0) = E_1$. This requires the full 6×6 E matrix.

$$E(t_n) = \begin{bmatrix} \overline{\underline{e}_r\underline{e}_r^T} & \overline{\underline{e}_r\underline{e}_v^T} \\ \overline{\underline{e}_v\underline{e}_r^T} & \overline{\underline{e}_v\underline{e}_v^T} \end{bmatrix}$$

$E_2 =$ upper left 3×3 partition of $E(t_n)$



For small times around t_n,

$$\underline{e}(t) = \underline{e}(t_n) + \underline{v}(t_n)(t - t_n)$$

$$= \underline{e}(t_n) + (\underline{v}_n(t_n) + \underline{e}_v(t_n))(t - t_n)$$

$$= \underline{e}(t_n) + \underline{v}_n(t_n)(t - t_n)$$

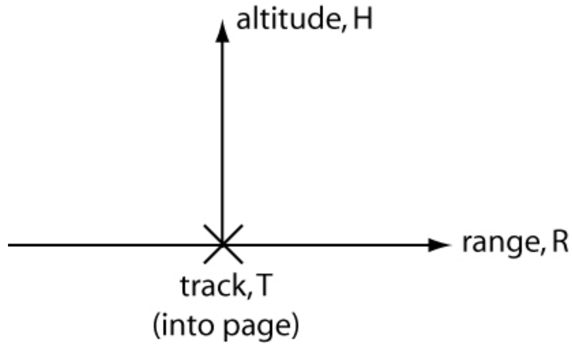
$$\underline{1}_v^T\underline{e}(t) = \underline{1}_v^T\underline{e}_2 + \underline{1}_v^T\underline{v}_n(t_n)(t - t_n) = 0$$

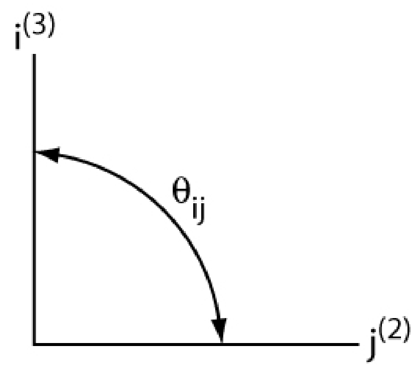
$$(t_i - t_n) = -\frac{\underline{1}_v^T\underline{e}_2}{\underline{1}_v^T\underline{v}_n}$$

$\underline{e}_3$ = position error at impact

$$= \underline{e}_2 - \underline{v}_n \frac{\underline{1}_v^T \underline{e}_2}{\underline{1}_v^T \underline{v}_n}$$

$$= \left[I - \frac{\underline{v}_n \underline{1}_v^T}{\underline{1}_v^T \underline{v}_n} \right] \underline{e}_2 \quad \text{"projection matrix"}$$





$$e_3' = [R]\underline{e}_2$$

$$R = \begin{bmatrix} \cos\theta_{ij} & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

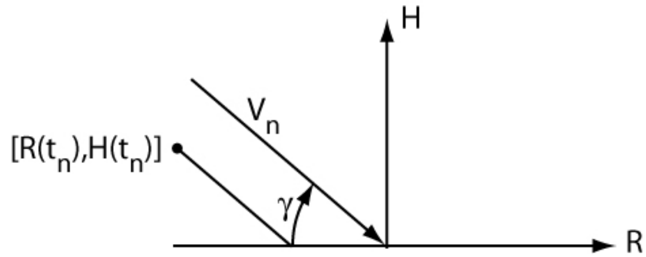
$$R_{ij} = \cos\theta_{ij}$$

$$R = \begin{bmatrix} \underline{1}_1 & \underline{1}_2 & \underline{1}_3 \end{bmatrix}$$

$\underline{1}_j$ **=** unit vectors along the j^{th} axis of the **2** frame expressed in the coordinates of the **3** frame.

$$\underline{e}_3' = R\underline{e}_2$$

$$E_3' = RE_2R^T$$



$$e_R(t) = e_{R_3} + v_n \cos\gamma (t - t_n)$$
$$e_T(t) = e_{T_3}$$
$$e_H(t) = e_{H_3} - v_n \sin\gamma (t - t_n)$$

Impact:

$$e_H(t_i) = e_{H_3} - v_n \sin\gamma (t_i - t_n) = 0$$
$$(t_i - t_n) = \frac{1}{v_n \sin\gamma} e_{H_3}$$
$$e_R(t_i) = e_{R_3} + \frac{v_n \cos\gamma}{v_n \sin\gamma} e_{H_3}$$
$$= e_{R_3} + \cot\gamma e_{H_3}$$
$$e_T(t_i) = e_{T_3}$$

The transformation which relates R,H,T errors at the nominal end time to R and T errors when H=0 is:

$$\underline{e}_4 = \begin{bmatrix} e_R(t_i) \\ e_T(t_i) \end{bmatrix}$$
$$= \begin{bmatrix} e_{R_3} + \cot\gamma e_{H_3} \\ e_{T_3} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & \cot\gamma \\ 0 & 1 & 0 \end{bmatrix} \underline{e}_3' \equiv P\underline{e}_3'$$

If the $\underline{e}_s$ defined earlier, based on integration of perturbed trajectories, is measured in R,T coordinates, then the sensitivity matrix defined at that point is equivalent to

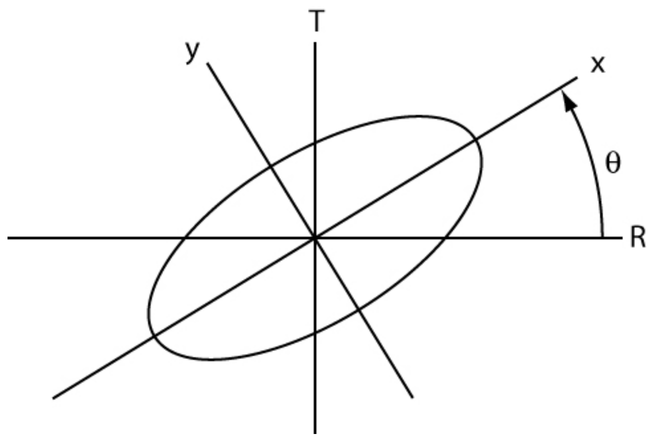
$$\underline{e}_s = S\underline{e}_1$$
$$S = PR\Phi_r$$

$$E_4 = PE_3'P^T$$
$$= \begin{bmatrix} \overline{R^2} & \overline{RT} \\ \overline{TR} & \overline{T^2} \end{bmatrix} = \begin{bmatrix} \sigma_R^2 & \mu_{RT} \\ \mu_{RT} & \sigma_T^2 \end{bmatrix} = \begin{bmatrix} \sigma_R^2 & \rho\sigma_R\sigma_T \\ \rho\sigma_R\sigma_T & \sigma_T^2 \end{bmatrix}$$

If all the original error sources are assumed normal, *R* and *T* will have a joint binormal distribution since they are derived from the error sources by linear operations only. This joint probability density function is

$$f(r,t) = \frac{1}{2\pi\sigma_R\sigma_T\sqrt{1-\rho^2}} e^{-\left[\frac{\left(\frac{r}{\sigma_R}\right)^2 - 2\rho\left(\frac{r}{\sigma_R}\right)\left(\frac{t}{\sigma_T}\right) + \left(\frac{t}{\sigma_T}\right)^2}{2(1-\rho^2)}\right]}$$

where σ_R, σ_T and ρ can be identified from E_4. Recall that we are considering unbiased errors.



Contour of constant probability density function is

$$\left(\frac{r}{\sigma_R}\right)^2 - 2\rho\left(\frac{r}{\sigma_R}\right)\left(\frac{t}{\sigma_T}\right) + \left(\frac{t}{\sigma_T}\right)^2 = c^2$$
$$r = x\cos\theta - y\sin\theta$$
$$t = x\sin\theta + y\cos\theta$$

Get:

$$(\theta)x^2 + \underbrace{(\theta)}_{=0}xy + (\theta)y^2 = c^2$$

Coefficient of x, y equals zero for principal axes.

$$\tan 2\theta = \frac{2\rho\sigma_R\sigma_T}{\sigma_R^2 - \sigma_T^2} = \frac{2\mu_{RT}}{\sigma_R^2 - \sigma_T^2}$$

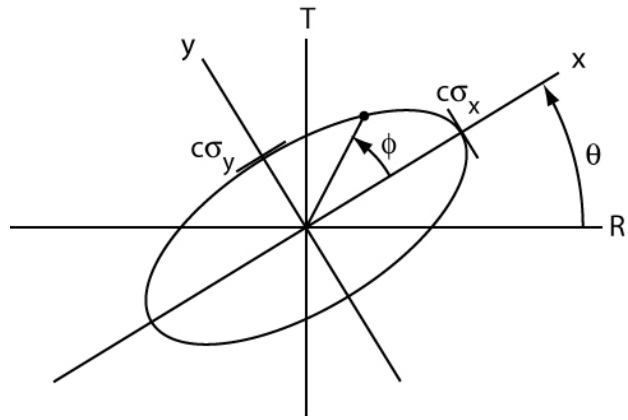
Use a 4 quadrant $\tan^{-1}$ function.

Once θ is found, can plug into pdf expression, get σ_x and σ_y.

$$h = \sqrt{(\sigma_R^2 - \sigma_T^2)^2 + (2\rho\sigma_R\sigma_T)^2}$$

$$\sigma_x^2 = \frac{1}{2}(\sigma_R^2 + \sigma_T^2 + h)$$

$$\sigma_y^2 = \frac{1}{2}(\sigma_R^2 + \sigma_T^2 - h)$$



$$x_i = c\sigma_x \cos\phi_i$$

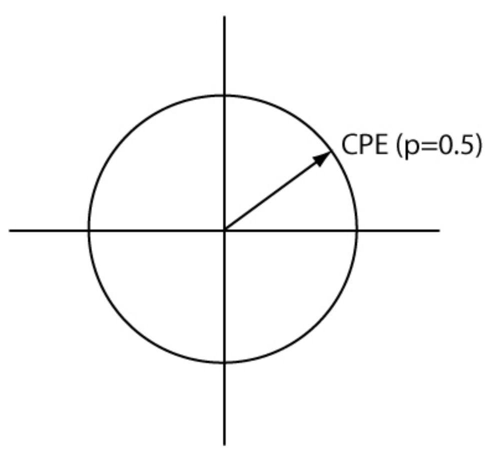
$$y_i = c\sigma_y \sin\phi_i$$

May want to choose c to achieve a certain probability of lying in that contour.

In principal coordinates, the probability of a point inside a "$c\sigma$" ellipse is

$$P = 1 - e^{-\frac{c^2}{2}}$$

People often choose c to find what is called the circular probable error (CPE).



$$\sigma = \frac{1}{2}(\sigma_x + \sigma_y)$$

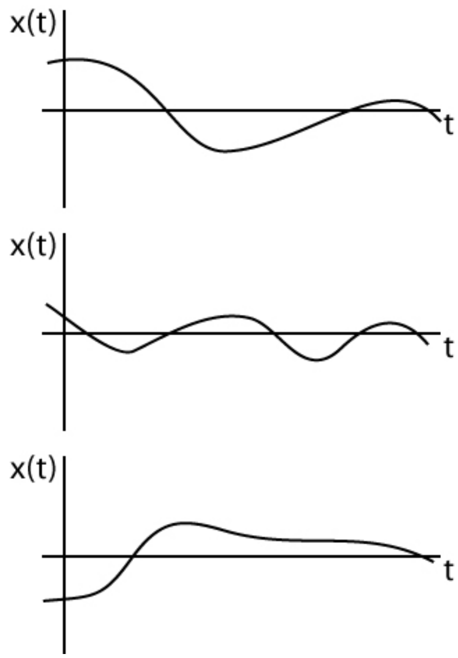
Choosing P=0.5, c=1.177

$$CPE = 0.588(\sigma_x + \sigma_y)$$

This approximation is good to an ellipticity of around 3.

Random Processes

A random process is an ensemble of functions of time which occur at random.



In most instances we have to imagine a non-countable infinity of possible functions in the ensemble.

There is also a probability law which determines the chances of selecting the different members of the ensemble.

We generally characterize random processes only partially.

One important descriptor – the first order distribution.

This is the classical description of random processes. We will also give the state space description later.

$x(t_1)$ is a random variable.

$F(x,t) = P\left[x(t) \le x\right]$, where $x(t)$ is the name of a process and x is the value taken

$$f(x,t) = \frac{dF(x,t)}{dx}$$