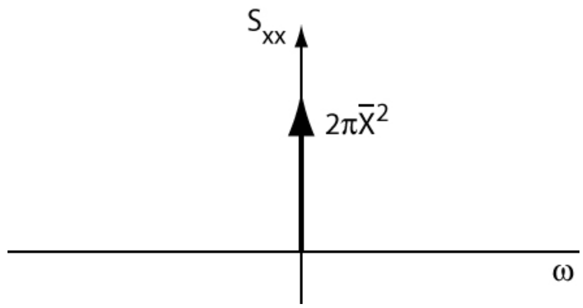
Lecture 12

Non-zero power at zero frequency



Non-zero power at non-zero frequency

If $R_{xx}(\tau)$ includes a sinusoidal component corresponding to the component
$x(t) = A\sin(\omega_0 t + \theta)$
where θ is uniformly distributed over 2π, A is random, independent of θ, that component will be

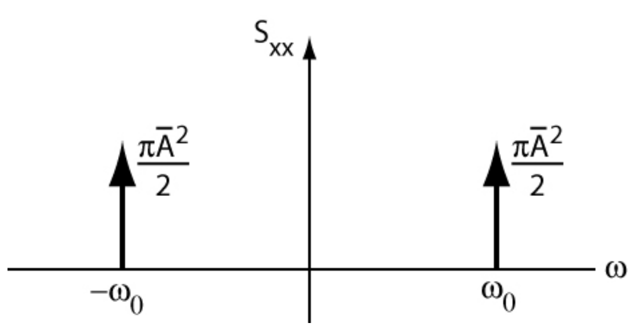
$$R_{xx}(\tau) = \frac{1}{2}\overline{A^2}\cos\omega_0\tau$$

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} \frac{1}{2}\overline{A^2}\cos\omega_0\tau e^{-j\omega\tau}d\tau$$

$$= \frac{1}{2}\overline{A^2}\int_{-\infty}^{\infty} \frac{1}{2}\left[e^{j\omega_0\tau} + e^{-j\omega_0\tau}\right]e^{-j\omega\tau}d\tau$$

$$= \frac{1}{2}\overline{A^2}\int_{-\infty}^{\infty} \frac{1}{2}\left[e^{j(\omega-\omega_0)\tau} + e^{-j(\omega+\omega_0)\tau}\right]d\tau$$

$$= \frac{1}{2}\pi\overline{A^2}\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$



Units of S_{xx}

Mean squared value per unit frequency interval.

Usually:

$$S_{xx}(\omega) = \int_{\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$$

$$\overline{x^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega$$

$$= \int_{-\infty}^{\infty} S_{xx}(f) df, \text{ where } f = \frac{\omega}{2\pi}$$

In this case, $S_{xx} \sim \frac{q^2}{Hz}$

Next most common:

$$S_{xx}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$$

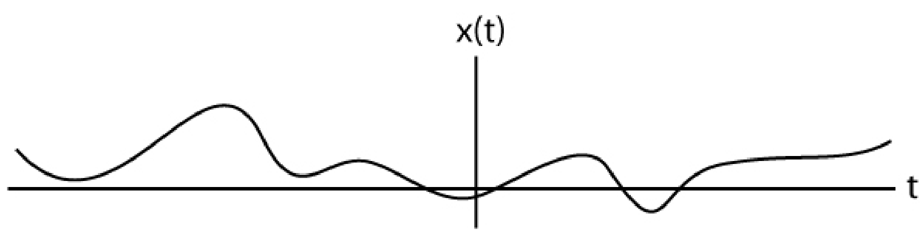
$$\overline{x^2} = \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega$$

In this case, $S_{xx} \sim \frac{q^2}{\text{rad/sec}} = q^2 \text{ sec}$

There is an alternate form of the power spectral density function.

Since $S_{xx}(\omega)$ is a measure of the power density of the harmonic components of $x(t)$, one should be able to get $S_{xx}(\omega)$ also from the Fourier Transform of $x(t)$ which is a direct decomposition of $x(t)$ into its infinitesimal harmonic components.

This is true, and is the approach taken in the text. One difficulty is that the Fourier Transform does not converge for members of stationary ensembles. The mathematics are handled by a limiting process.



Define $x_T(t) = \begin{cases} x(t), & (-T < t < T) \\ 0, & \text{elsewhere} \end{cases}$

$$X_T(\omega) = \int_{-\infty}^{\infty} x_T(t) e^{-j\omega\tau} d\tau$$

$$= \int_{-T}^{T} x(t) e^{-j\omega t} dt$$

Then

$$S_{xx}(\omega) = \lim_{T\to\infty} \frac{\overline{X_T(\omega)^* X_T(\omega)}}{2T}$$

$$= \lim_{T\to\infty} \frac{\overline{|X_T(\omega)|^2}}{2T}$$

Notice that the operations of transforming and averaging and product are done in opposite order here than if the transform of the autocorrelation function is calculated.

If one has only a finite record of a single random function, and $S_{xx}(\omega)$ is to be so calculated approximately under the ergodic hypothesis, it can be done either way.

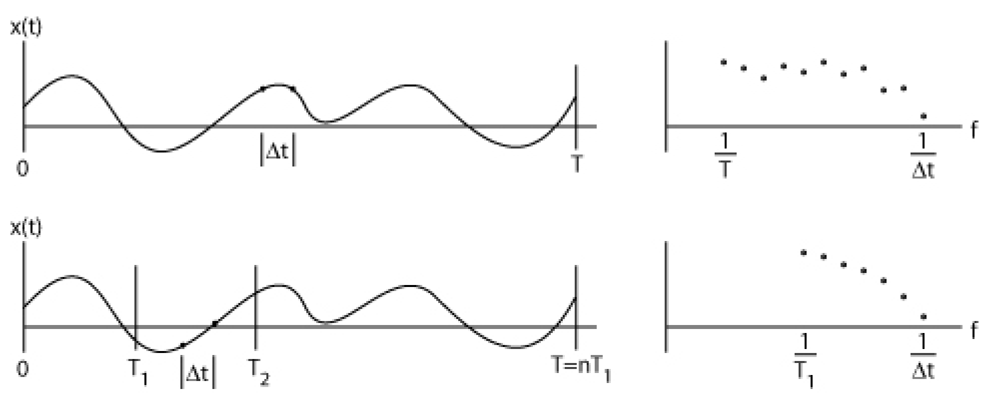
$$R_{xx}(\tau) = \frac{2}{T-\tau} \int_0^{T-\tau} x(t)x(t+\tau) d\tau$$

$$S_{xx}(\omega) = 2\int_0^{\tau_{\max}} R_{xx}(\tau) \cos\omega\tau d\tau$$

$$x(\omega) = \int_{-T}^{T} x(t) e^{-j\omega t} dt$$

$$S_{xx}(\omega) = \frac{X(\omega)^* X(\omega)}{2T}$$

Standard deviation of S_{xx} measured this way is approximately equal to mean.



The second approach is faster.

In fact, with the advent of the Fast Fourier Transform (initiated by Cooley and Tukey), even if one wanted to calculate $R_{xx}(\tau)$ from $x(t)$, it is faster to transform $x(t)$ to $X(\omega)$, form $S_{xx}(\omega)$, and transform to get $R_{xx}(\tau)$ than to integrate $x(t)x(t+\tau)$ directly for all desired values of τ.

The Fast Fourier Transform is an amazingly efficient procedure for digital calculation of finite Fourier Transforms.

References:

Full issue – IEEE Transactions on Audio and Electroacoustics, Vol. AU-15, No.2; June 1967.
Tutorial article: Brighton, E.O. and Morrow, R.E.: The Fast Fourier Transform, IEEE Spectrum; Dec. 1967.

Cross spectral density

In dealing with more than one random process, the *cross power spectral density* arises naturally. For example, if

$$z(t) = x(t) + y(t)$$

where $x(t)$ and $y(t)$ are members of random ensembles, then we found before that

$$R_{zz}(\tau) = R_{xx}(\tau) + R_{xy}(\tau) + R_{yx}(\tau) + R_{yy}(\tau)$$

so that

$$S_{zz}(\omega) = \int_{-\infty}^{\infty} R_{zz}(\tau) e^{-j\omega\tau} d\tau$$

$$= S_{xx}(\omega) + S_{xy}(\omega) + S_{yx}(\omega) + S_{yy}(\omega)$$

where we have defined *cross spectral density functions*:

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j\omega\tau} d\tau$$

$$S_{yx}(\omega) = \int_{-\infty}^{\infty} R_{yx}(\tau) e^{-j\omega\tau} d\tau$$

This is equivalent to the definition

$$S_{xy}(\omega)=\lim_{T\to\infty}\frac{\overline{X_T(\omega)^* Y_T(\omega)}}{2T}$$

$$S_{yx}(\omega)=\lim_{T\to\infty}\frac{\overline{Y_T(\omega)^* X_T(\omega)}}{2T}$$

We note from this that:

$$S_{yx}(\omega)=S_{xy}(\omega)^*$$

so that the <u>sum of these two</u> as they appear in $S_{zz}(\omega)$ is <u>real</u>.

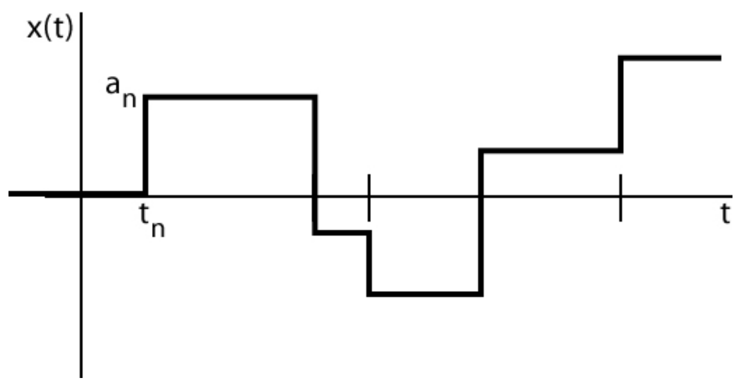
Also note that $S_{xy}(-\omega)=S_{xy}(\omega)^*=S_{yx}(\omega)$.

<u>Examples of Random Processes Analytically Defined</u>

Example: Random step function

Amplitude a_n independent, random

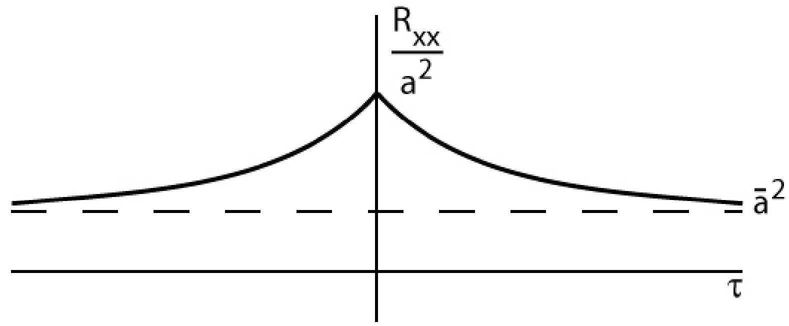
Change points t_n, Poisson-distributed with average density λ (points per second)



$$P(k)=\frac{1}{k!}\left(\lambda|\tau|\right)^k e^{-\lambda|\tau|}$$

$$P(0)=e^{-\lambda|\tau|}$$

$$\begin{aligned}R_{xx}(\tau)&=E\left[x(t)x(t+\tau)\right]\\&=P(\text{at least one change point in }|\tau|)\bar{a}^2+P(\text{no change point in }|\tau|)\overline{a^2}\\&=\left(1-e^{-\lambda|\tau|}\right)\bar{a}^2+e^{-\lambda|\tau|}\overline{a^2}\\&=\sigma_a^2e^{-\lambda|\tau|}+\bar{a}^2\end{aligned}$$



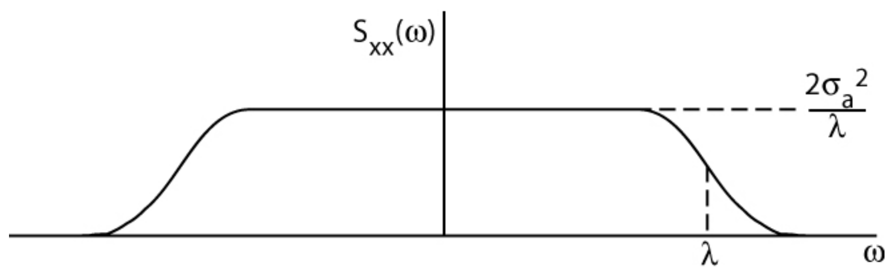
$$S_{xx}(\omega) = \int_{-\infty}^{\infty} \left[\bar{a}^2 + \sigma_a^2 e^{-\lambda|\tau|}\right] e^{-j\omega\tau} d\tau$$

$$= 2\pi\bar{a}^2\delta(\omega) + \int_{-\infty}^{0} \sigma_a^2 e^{-j\omega\tau} e^{\lambda\tau} d\tau + \int_{0}^{\infty} \sigma_a^2 e^{-j\omega\tau} e^{-\lambda\tau} d\tau$$

$$= 2\pi\bar{a}^2\delta(\omega) + \frac{\sigma_a^2}{\lambda - j\omega} e^{(\lambda - j\omega)\tau}\Big|_{-\infty}^{0} + \frac{\sigma_a^2}{-\lambda - j\omega} e^{-(\lambda + j\omega)\tau}\Big|_{0}^{\infty}$$

$$= 2\pi\bar{a}^2\delta(\omega) + \sigma_a^2\left[\frac{1}{\lambda - j\omega} + \frac{1}{\lambda + j\omega}\right]$$

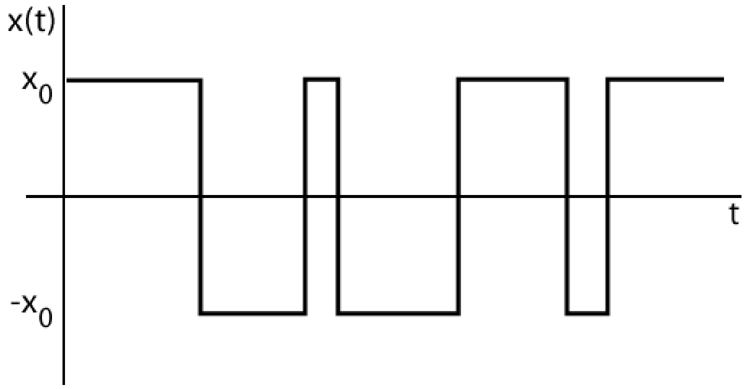
$$= 2\pi\bar{a}^2\delta(\omega) + \frac{2\sigma_a^2\lambda}{\lambda^2 + \omega^2}$$



Example: A signal random process

Reference: Newton, G.C, L.A. Gould and J.F. Kaiser. *Design of Linear Feedback Controls*. John Wiley, 1961. p.100. Papoulis calls this the semirandom telegraph signal; p.288.

A signal takes the values plus and minus x_0 only. It switches from one level to the other at "event points" which are Poisson distributed in time with constant average frequency λ. This is sometimes called a *telegraph signal.*



Average rate of occurrence of change points $=\lambda$ points per second

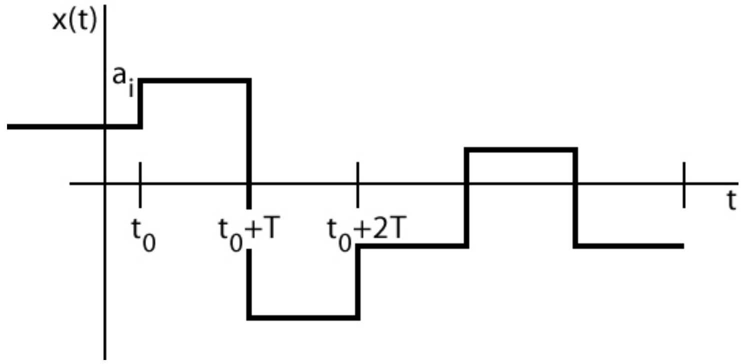
$$
\begin{aligned}
R_{xx}(\tau) &= E\left[x(t)x(t+|\tau|)\right] \\
&= P(\text{even number of change points in the interval } |\tau|)x_0^2 \\
&\quad +P(\text{odd number of change points in the interval } |\tau|)(-x_0^2) \\
&= x_0^2 \sum_{k=0,2,4,\ldots} \frac{1}{k!}\left(\lambda|\tau|\right)^k e^{-\lambda|\tau|} - x_0^2 \sum_{k=1,3,5,\ldots} \frac{1}{k!}\left(\lambda|\tau|\right)^k e^{-\lambda|\tau|} \\
&= \sum_{k=0,1,2,\ldots} \frac{1}{k!}(-1)^k\left(\lambda|\tau|\right)^k e^{-\lambda|\tau|} x_0^2 \\
&= x_0^2 e^{-\lambda|\tau|} \sum_{k=0}^{\infty} \frac{\left(-\lambda|\tau|\right)^k}{k!} \\
&= x_0^2 e^{-2\lambda|\tau|}
\end{aligned}
$$

You are generating higher harmonics in this case, as at each change point the amplitude changes sign. In the previous example, changes in amplitude at change point may be far smaller.

$$S_{xx}(\omega) = \frac{2(2)\lambda x_0^2}{\omega^2 + \left(2\lambda\right)^2}$$

Example: Binary function with an arbitrary amplitude distribution

The problem considers a binary function with a more general amplitude distribution. The distribution of a is restricted to ± 1 with equal probability. That gives the pseudo-random binary code used by GPS.



t_0 is uniformly distributed over $(0,T)$
Change points are periodic with period T
Amplitudes independent with $\bar{a},\overline{a^2}$

$$
\begin{aligned}
R_{xx}(\tau) &= E\left[x(t)x(t+\tau)\right] \\
&= P(\text{1 or more change points in } |\tau|)\bar{a}^2 + P(\text{No change point in } |\tau|)\overline{a^2} \\
&= \frac{|\tau|}{T}\bar{a}^2 + \left(1-\frac{|\tau|}{T}\right)\overline{a^2}, \quad |\tau| \le T
\end{aligned}
$$

$$
R_{xx}(\tau) = \begin{cases} \sigma_a^2\left(1-\dfrac{|\tau|}{T}\right)+\bar{a}^2, & |\tau| \le T \\ \bar{a}^2, & |\tau| > T \end{cases}
$$