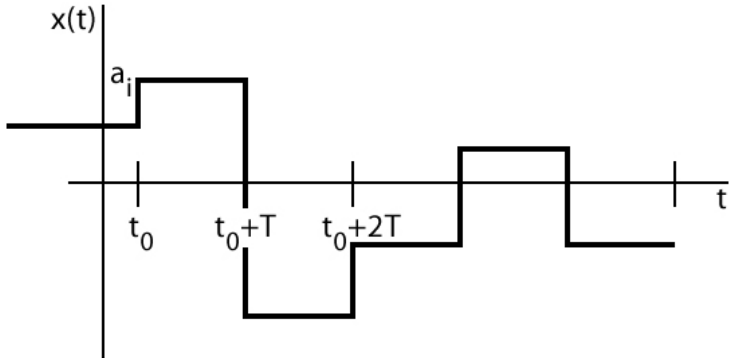
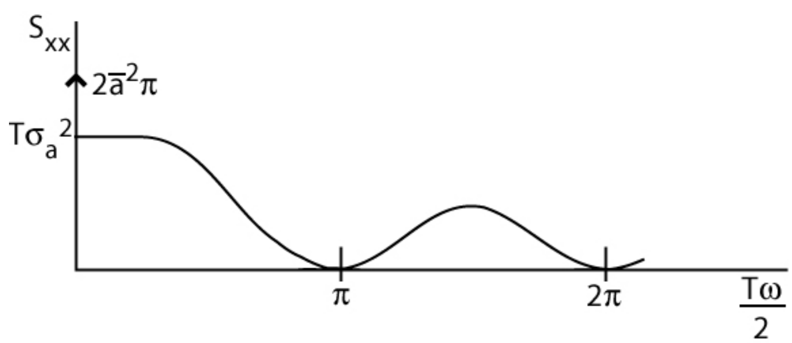
Lecture 13

Last time:

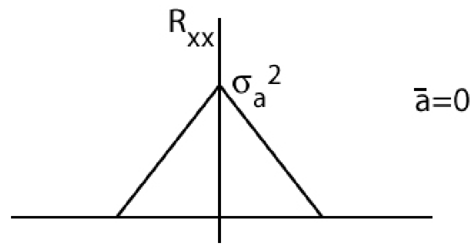


$$R_{xx}(\tau) = \begin{cases} \bar{a}^2 + \left(1 - \frac{|\tau|}{T}\right)\sigma_a^2, & (|\tau| \le T) \\ \bar{a}^2, & (|\tau| > T) \end{cases}$$

$$\begin{aligned} S_{xx}(\omega) &= \int_{-\infty}^{\infty} \bar{a}^2 e^{-j\omega\tau} d\tau + \int_{-T}^{T} \sigma_a^2 \left(1 - \frac{|\tau|}{T}\right) e^{-j\omega\tau} d\tau \\ &= 2\pi\bar{a}^2\delta(\omega) + 2\sigma_a^2 \int_0^T \left(1 - \frac{\tau}{T}\right)\cos\omega\tau d\tau \\ &= 2\pi\bar{a}^2\delta(\omega) + \frac{2\sigma_a^2}{T\omega^2}\left(1 - \cos T\omega\right) \\ &= 2\pi\bar{a}^2\delta(\omega) + T\sigma_a^2 \left(\frac{\sin\left(\frac{T\omega}{2}\right)}{\left(\frac{T\omega}{2}\right)} \right)^2 \end{aligned}$$

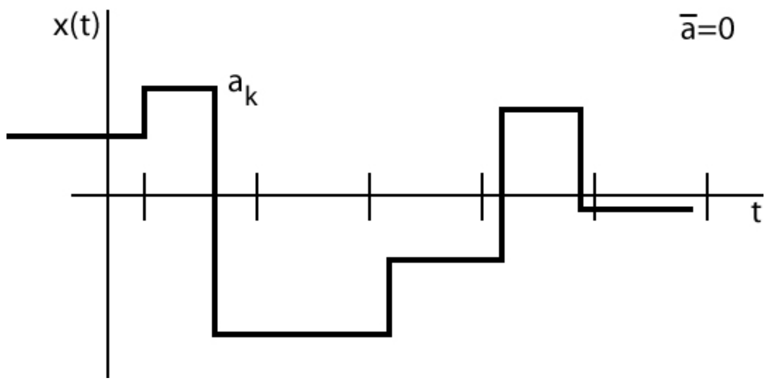


Amplitude of S_{xx} falls off, but not very rapidly.



Use error between early and late indicator to lock onto signal. Error is a linear function of shift, within the range $(-T, T)$.

Return to the 1st example process and take the case where the change points are Poisson distributed.



$$S_{xx}(\omega) = \frac{2\lambda\sigma_a^2}{\omega^2 + \lambda^2}$$

Take the limit of this as σ_a^2 and λ become large in a particular relation: to establish the desired relation, replace

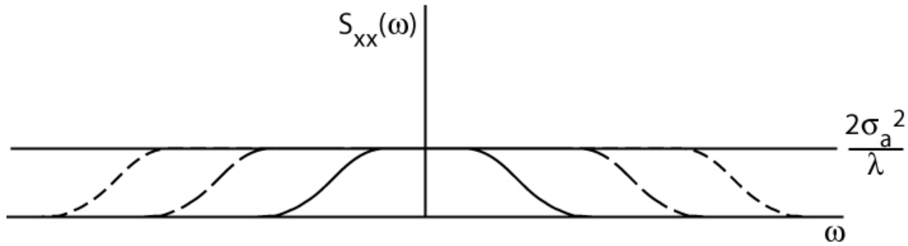
$$\sigma_a^2 \to k\sigma_a^2$$
$$\lambda \to k\lambda$$

$$S_{xx}(\omega) = \frac{2k\lambda k\sigma_a^2}{\omega^2 + (k\lambda)^2}$$

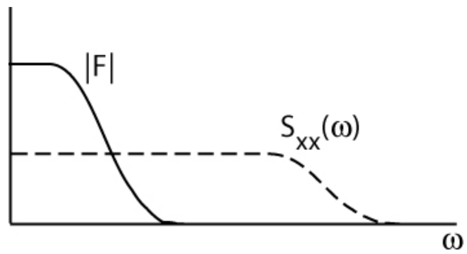
and take the limit as $k \to \infty$.

$$\lim_{k\to\infty} S_{xx}(\omega) = \lim_{k\to\infty} \frac{2k^2\lambda\sigma_a^2}{(k\lambda)^2} = \frac{2\sigma_a^2}{\lambda}$$

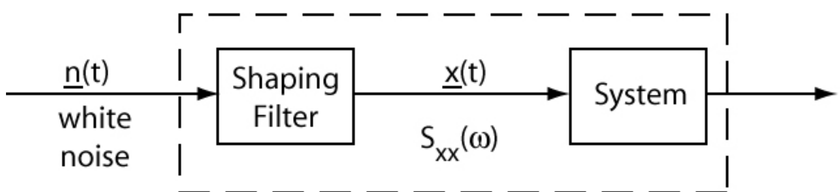
Note this is independent of frequency.



This is defined to be a "white noise" by analogy with white light, which is supposed to have equal participation by all wavelengths.



Can shape $x(t)$ to the correct spectrum so that it can be analyzed in this manner, by adding a shaping filter in the state-space formulation.



Definition of a white noise process

White means constant spectral density.

$$S_{xx}(\omega) = S_0, \quad \text{constant}$$

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_0 e^{j\tau\omega} d\omega$$

$$= S_0 \delta(\tau)$$

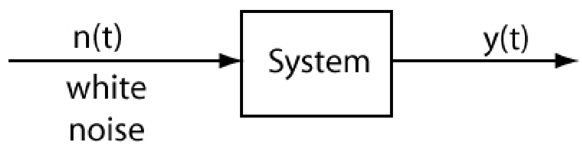
White noise processes only have a defined power density. The variance of a white noise process is not defined.

If you start with almost any process $x(t)$,

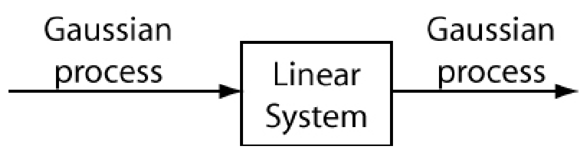
$\lim_{a\to\infty} \sqrt{a}\, x(at) \Rightarrow$ is a white noise

So far we have looked only at the R_{xx} and S_{xx} of processes. We shall find that if we wish to determine only the R_{yy} or S_{yy} (thus the $\overline{y^2}$) of outputs of linear systems, all we need to know about the inputs are their R_{xx} or S_{xx}.

But what if we wanted to know the probability that the error in a dynamic system would exceed some bound? For this we need the first probability density function of the system error – an output. Very difficult in general.



The pdf of the output $y(t)$ satisfies the Fokker-Planck partial differential equation – also called the Kolmagorov forward equation. Applies to a continuous dynamic system driven by a white noise process.



One case is easy: Gaussian process into a linear system, output is Gaussian.

Gaussian processes are defined by the property that probability density functions of all order are normal functions.

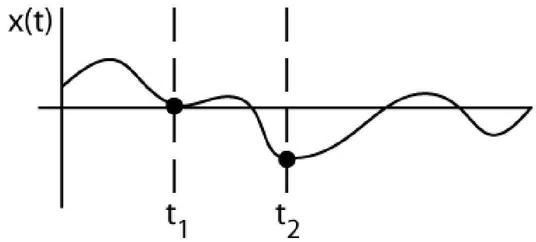
$$f_n(x_1,t_1;x_2,t_2;...x_n,t_n) = n\text{-dimensional normal}$$

$$f_n(\underline{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{|M|}} e^{-\frac{1}{2}\underline{x}^T M^{-1}\underline{x}}$$

M is the covariance matrix for

$$\underline{x} = \begin{bmatrix} x(t_1) \\ x(t_2) \\ \vdots \\ x(t_n) \end{bmatrix}$$

Thus, $f_n(\underline{x})$ for all n is determined by M - the covariance matrix for $\underline{x}$.



$$M_{ij} = \overline{x(t_i)x(t_j)}$$
$$= R_{xx}(t_i, t_j)$$

Thus for a Gaussian process, the autocorrelation function completely defines all the statistical properties of the process since it defines the probability density functions of all order. This means:

If $R_{yx}(t_i, t_j) = R_{xx}\left(\left|t_j - t_i\right|\right)$, the process is <u>stationary</u>.

If two processes $x(t), y(t)$ are jointly Gaussian, and are uncorrelated $\left(R_{xy}(t_i, t_j) = 0\right)$, they are <u>independent</u> processes.

Most important: Gaussian input → linear system → Gaussian output. In this case all the statistical properties of the output are determined by the correlation function of the output – for which we shall require only the correlation functions for the inputs.

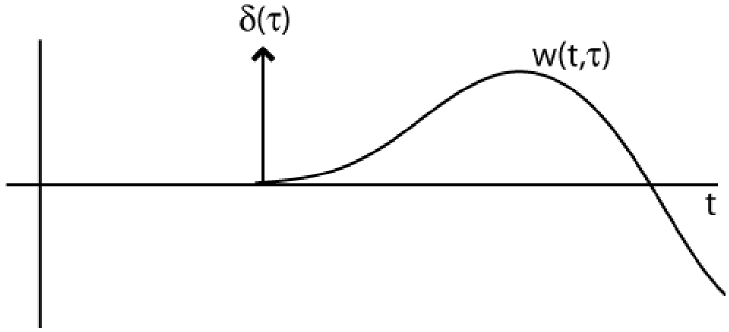
Upcoming lectures <u>will not</u> cover several sections that deal with:
Narrow band Gaussian processes
Fast Fourier Transform
Pseudorandom binary coded signals

These are important topics for your general knowledge.

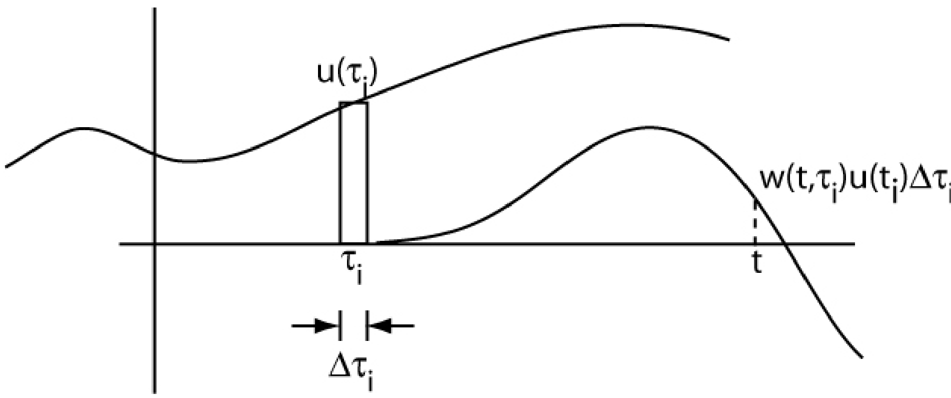
Characteristics of Linear Systems

Definition of linear system:
If
$u_1(t) \rightarrow y_1(t)$
$u_2(t) \rightarrow y_2(t)$
Then
$au_1(t) + bu_2(t) \rightarrow ay_1(t) + by_2(t)$

By characterizing the response to a standard input, the response to any input can be constructed by superposing responses using the system's linearity.



$w(t,\tau)$ is the weighting function.



$$y(t) = \lim_{\Delta\tau_i \to 0} \sum_i w(t,\tau_i)u(\tau_i)\Delta\tau_i \to \int_{-\infty}^{t} w(t,\tau)u(\tau)d\tau$$

The central limit theorem says $y(t) \to$ normal if $u(t)$ is white noise.

Stable if every bounded input gives rise to a bounded output.

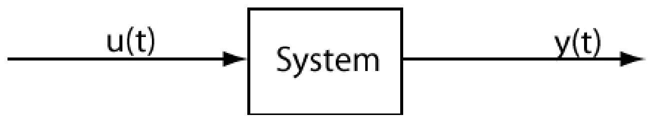
$$\int_{-\infty}^{\infty} |w(t,\tau| d\tau = \text{const.} < \infty \quad \Rightarrow \text{bounded for all } t$$

Realizable if causal.

$$w(t,\tau) = 0, \quad (t < \tau)$$

State Space – An alternate characterization for a linear differential system

If the input and output are related by an nth order linear differential equation, once can also relate input to output by a set of n linear first order differential equations.



$$\underline{\dot{x}}(t) = A(t)\underline{x}(t) + B(t)\underline{u}(t)$$
$$\underline{y}(t) = C(t)\underline{x}(t)$$

The solution form is:

$$\underline{y}(t) = C(t)\underline{x}(t)$$
$$\underline{x}(t) = \Phi(t,t_0)\underline{x}(t_0) + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)\underline{u}(\tau)d\tau$$

where $\Phi(t,\tau)$ satisfies:

$$\frac{d}{dt}\Phi(t,\tau) = A(t)\Phi(t,\tau), \qquad \Phi(\tau,\tau) = I$$

Note that any system which can be cast in this form is not only mathematically realizable but practically realizable as well. Must add a gain times $\underline{u}$ to $\underline{y}$ to get as many zeroes as poles.

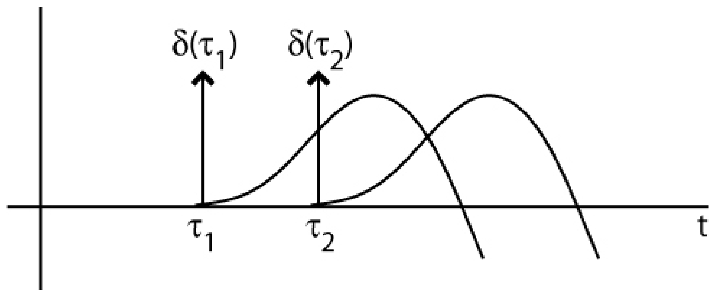
For comparison with the weighting function description, take $\underline{u}$ and $\underline{y}$ to be scalars, and take $t_0 = -\infty$. For stable systems, the transition from $-\infty$ to any finite time is zero.

Specialize the state space model to single-input, single-output (SISO) and $t_0 \to -\infty$:

$$\Phi(t,-\infty) = 0$$
$$y(t) = \underline{C}(t)^T \underline{x}(t)$$
$$\underline{x}(t) = \int_{-\infty}^{t} \Phi(t,\tau)\underline{b}(\tau)u(\tau)d\tau$$
$$y(t) = \int_{-\infty}^{t} \underline{C}(t)^T \Phi(t,\tau)\underline{b}(\tau)u(\tau)d\tau$$
$$w(t,\tau) = \underline{C}(t)^T \Phi(t,\tau)\underline{b}(\tau)$$

which we recognize by comparison with the earlier expression for $y(t)$.

For an invariant system, the shape of $w(t,\tau)$ is independent of the time the input was applied; the output depends only on the elapsed time since the application of the input.



$w(t,\tau) \rightarrow w(t-\tau) = w(t), \quad \tau = 0$ by convention

Stability:

$$\int_{-\infty}^{\infty} \left| w(t-\tau) \right| d\tau = \int_{-\infty}^{\infty} \left| w(t) \right| dt = \text{const.} < \infty$$

Note this implies $w(t)$ is Fourier transformable.

Realizability:

$w(t-\tau) = 0, \quad (t < \tau)$

$w(t) = 0, \quad (t < 0)$