Lecture 20

Last time: Completed solution to the optimum linear filter in real-time operation

Semi-free configuration:

$$H_0(s) = \frac{1}{2\pi j F(s)_L F(-s)_L S_{ii}(s)_L} \int_0^\infty dt e^{-st} \int_{-j\infty}^{j\infty} dp \underbrace{\frac{D(p)F(-p)_L S_{is}(p)}{F(p)_R S_{ii}(p)_R}}_{\left[(p)\right]} e^{pt}$$

Special case: $\left[(p)\right]$ is rational:

In this solution formula we can carry out the indicated integrations in literal form in the case in which $\left[(p)\right]$ is rational.

In our work, we deal in a practical way only with rational F, S_{is}, and S_{ii}, so this function will be rational if $D(p)$ is rational. This will be true of every desired operation except a predictor. Thus except in the case of prediction, the above function which will be symbolized as $[\ ]$ can be expanded into

$$\left[(p)\right] = \left[(p)\right]_L + \left[(p)\right]_R$$

where $[\ ]_L$ has poles only in LHP and $[\ ]_R$ has poles only in RHP. The zeroes may be anywhere.

For rational $[\ ]$, this expansion is made by expanding into partial fractions, then adding together the terms defining LHP poles to form $[\ ]_L$ and adding together the terms defining RHP poles to form $[\ ]_R$. Actually, only $[\ ]_L$ will be required.

$$\frac{1}{2\pi j} \int_0^\infty dt e^{-st} \int_{-j\infty}^{j\infty} dp \left\{ \left[(p)\right]_L + \left[(p)\right]_R \right\} e^{pt} = \int_0^\infty f_L(t) e^{-st} dt + \int_0^\infty f_R(t) e^{-st} dt$$

where

$$f_L(t) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left[(p)\right]_L e^{pt} dp = 0,\ t < 0$$

$$f_R(t) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left[(p)\right]_R e^{pt} dp = 0,\ t > 0$$

Note that $f_R(t)$ is the inverse transform of a function which is analytic in LHP; thus $f_R(t) = 0$ for $t > 0$ and

$$\int_0^{\infty} f_R(t)e^{-st}dt = 0$$

Also $f_L(t)$ is the inverse transform of a function which is analytic in RHP; thus $f_L(t) = 0$ for $t < 0$. Thus

$$\int_0^{\infty} f_L(t)e^{-st}dt = \int_{-\infty}^{\infty} f_L(t)e^{-st}dt = \left[(s)\right]_L$$

Thus finally,

$$H_0(s) = \frac{\left[\dfrac{D(s)F(-s)_L S_{ij}(s)}{F(s)_R S_{ii}(s)_R}\right]_L}{F(s)_L F(-s)_L S_{ii}(s)_L}$$

In the usual case, $F(s)$ is a stable, minimum phase function. In that case, $F(s)_L = F(s)$, $F(s)_R = 1$; that is, all the poles and zeroes of $F(s)$ are in the LHP. Similarly, $F(-s)_L = 1$. Then

$$H_0(s) = \frac{\left[\dfrac{D(s)S_{ij}(s)}{S_{ii}(s)_R}\right]_L}{F(s)S_{ii}(s)_L}$$

Thus in this case the optimum transfer function from input to output is

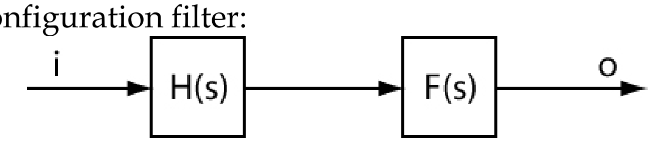
$$F(s)H_0(s) = \frac{\left[\dfrac{D(s)S_{ij}(s)}{S_{ii}(s)_R}\right]_L}{S_{ii}(s)_L}$$

and the optimum function to be cascaded with the fixed part is obtained from this by division by $F(s)$, so that the fixed part is compensated out by cancellation.

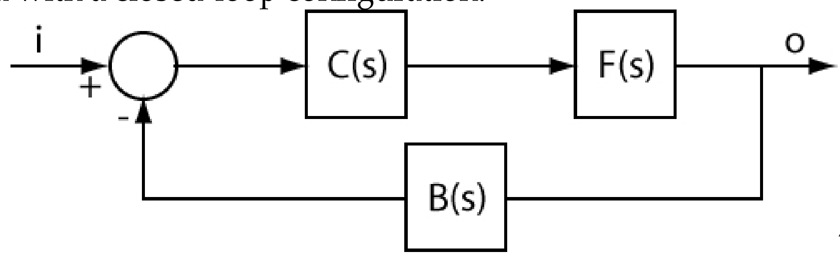
Free configuration problem:

$$H_0(s) = \frac{\left[\dfrac{D(s)S_{ij}(s)}{S_{ii}(s)_R}\right]_L}{S_{ii}(s)_L}$$

Optimum free configuration filter:



We started with a closed-loop configuration:



"

$$H(s) = \frac{C(s)}{1 + C(s)F(s)B(s)}$$

$$C(s) = \frac{H(s)}{1 - H(s)F(s)B(s)}$$

The loop will be stable, but $C(s)$ may be unstable.

Special comments about the application of these formulae:

a) Unstable $F(s)$ cannot be treated because the Fourier transform of $w_F(t)$ does not converge in that case. To treat this system, first close a feedback loop around $F(s)$ to create a stable "fixed" part and work with this stable feedback system as $F(s)$. When the optimum compensation is found, it can be collected with the original compensation if desired.

b) An $F(s)$ which has poles on the $j\omega$ axis is the limiting case of functions for which the Fourier transform converges. You can move the poles just into the LHP by adding a real part $+\varepsilon$ to the pole locations. Solve the problem with this ε and at the end set it to zero.
Zeroes of $F(s)$ on $j\omega$ axis can be included in either factor and the result will be the same. This will permit cancellation compensation of poles of $F(s)$ on the $j\omega$ axis, including poles at the origin.

c) In factoring $S_{ii}(s)$ into $S_{ii}(s)_L S_{ii}(s)_R$, any constant factor in $S_{ii}(s)$ can be divided between $S_{ii}(s)_L$ and $S_{ii}(s)_R$ in any convenient way. The same is true of $F(s)$ and $F(-s)$.

d) Problems should be well-posed in the first place. Avoid combinations of $D(s)$ and $S_{ss}(\omega)$ which imply infinite $\overline{d(t)^2}$ because that may assume infinite $\overline{e^2}$ for any realizable filter. Such as a differentiator on a signal which falls off as $\frac{1}{\omega^2}$.

e) The point at $t = 0$ was left hanging in several steps of the derivation of the solution formula. Don't bother checking the individual steps; just check the final solution to see if it satisfies the necessary conditions.

The Wiener-Hopf equation requires $l(\tau_1)=0$ for $\tau_1 \ge 0$. Thus $L(s)$ should be analytic in LHP and go to zero at least as fast as $\frac{1}{s}$ for large $|s|$.

$$L(s) = F(-s)H_0(s)F(s)S_{ii}(s) - F(-s)D(s)S_{is}(s)$$

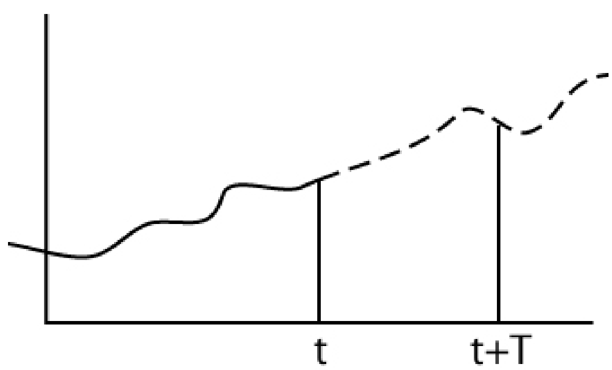
We have solved the problem of the optimum linear filter under the least mean squared error criterion.

Further analysis shows that if the inputs, signal and noise, are Gaussian, the result we have is the optimum filter. This is, there is no filter, linear or nonlinear which will yield smaller mean squared error.

If the inputs are not both Gaussian, it is almost sure that some nonlinear filters can do better than the Wiener filter. But theory for this is only beginning to be developed on an approximate basis.

Note that if we only know the second order statistics of the inputs, the optimum linear filter is the best we can do. To take advantage of nonlinear filtering we must know the distributions of the inputs.

Example: Free configuration predictor (real time)



$$S_{ss}(s) = \frac{A}{a^2 - s^2}$$

$$S_{nn}(s) = S_n$$

The s, n are uncorrelated.

$$D(s) = e^{sT}$$

Use the solution form

$$H_0(s) = \frac{1}{2\pi j S_{ii}(s)_L} \int_0^\infty dt e^{-st} \int_{-j\infty}^{j\infty} dp \frac{D(p) S_{is}(p)}{S_{ii}(p)_R} e^{pt}$$

$$= \frac{1}{2\pi j S_{ii}(s)_L} \int_0^\infty dt e^{-st} \int_{-j\infty}^{j\infty} dp \frac{S_{is}(p)}{S_{ii}(p)_R} e^{p(t+T)}$$

$$\begin{aligned} S_{ii}(s) &= S_{ss}(s) + S_{nn}(s) \\ &= \frac{A}{a^2+s^2} + S_n \\ &= S_n \left[\frac{\frac{A}{S_n} + a^2 - s^2}{a^2 - s^2} \right] \\ &= S_n \frac{b^2 - s^2}{a^2 - s^2} \\ &= \left[S_n \frac{b+s}{a+s} \right] \left[\frac{b-s}{a-s} \right] \end{aligned}$$

where $b^2 = a^2 + \frac{A}{S_n}$.

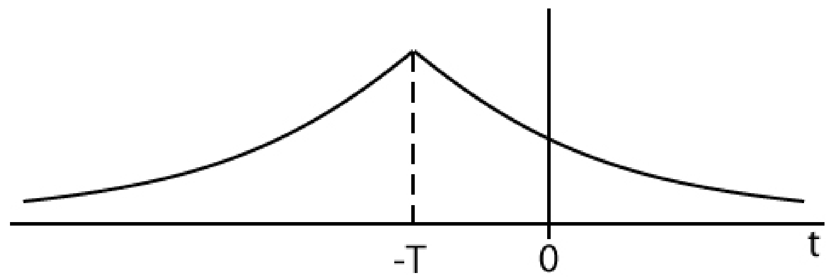
$$S_{is}(p) = S_{ss}(p) = \frac{A}{(a+s)(a-s)}$$

$$\begin{aligned} \frac{S_{is}(p)}{S_{ii}(p)_R} &= \frac{A(a-s)}{(a+s)(a-s)(b-s)} \\ &= \frac{A}{(a+s)(b-s)} \\ &= \frac{\frac{A}{b+a}}{a+s} + \frac{\frac{A}{a+b}}{b-s} \end{aligned}$$

Using the integral form,

$$\frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{\frac{A}{a+b}}{p+a} e^{p(t+T)} dp = \begin{cases} \frac{A}{a+b} e^{-a(t+T)}, \ t > -T \\ 0, \text{ otherwise} \end{cases}$$

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\frac{\frac{A}{a+b}}{b-p}e^{p(t+T)}dp=\begin{cases}\frac{A}{a+b}e^{b(t+T)}, & t<-T\\ 0, & \text{otherwise}\end{cases}$$



$$\int_0^\infty e^{-st}\frac{A}{a+b}e^{-a(t+T)}dT=\frac{A}{a+b}e^{-aT}\int_0^\infty e^{-(s+a)t}dt$$
$$=\frac{A}{a+b}e^{-aT}\frac{1}{s+a}$$

$$H_0(s)=\frac{A}{a+b}e^{-aT}\frac{(s+a)}{(s+a)S_n(s+b)}=\frac{Ae^{-aT}}{S_n(a+b)}\frac{1}{s+b}$$

where $b=\sqrt{a^2+\frac{A}{S_n}}$.

Note the bandwidth of this filter and the gain $\sim e^{-aT}$ which is the correlation between $S(t)$ and $S(t+T)$

Example: Semi-free problem with non-minimum phase F. Optimum compensator

$$F(s)=K\frac{(c-s)}{s(d+s)}$$

$$S_{ss}(s)=\frac{A}{a^2-s^2}$$

$$S_{nn}(s)=S_n$$

The s,n are uncorrelated.

Servo example where we'd like the output to track the input, so the desired operator, $D(s)=1$.

$$S_{ii}(s) = S_n \frac{b^2 - s^2}{a^2 - s^2}$$

$$S_{ii}(s)_L = S_n \frac{b + s}{a + s}$$

$$S_{ii}(s)_R = S_n \frac{b - s}{a - s}$$

$$F(s)_L = \frac{K}{(s+\varepsilon)(d+s)}$$

$$F(s)_R = c - s$$

$$F(-s) = K \frac{c+s}{(\varepsilon - s)(d - s)}$$

$$F(-s)_L = c + s$$

$$\left[(s)\right] = \frac{D(s)F(-s)_L S_{is}(s)}{F(s)_R S_{ii}(s)_R}$$

$$= \frac{(c+s)A(a-s)}{(a+s)(a-s)(c-s)(b-s)} = \frac{A(c+s)}{(a+s)(c-s)(b-s)}$$