9.07 Introduction to Statistics for Brain and Cognitive Sciences
Emery N. Brown

Hypothesis Testing Summary

1. Gaussian Distribution Unknown Mean with Known Variance

z-tests of One-Sided Alternatives

$H_0 : \mu = \mu_0$

A. $H_A : \mu > \mu_0$

Reject $H_0 : \mu = \mu_0$ with significance level α if

$$z = \frac{n^{\frac{1}{2}}(\bar{x} - \mu_0)}{\sigma} > z_{1-\alpha}$$

B. $H_A : \mu < \mu_0$

Reject $H_0 : \mu = \mu_0$ with significance level α if

$$z = \frac{n^{\frac{1}{2}}(\bar{x} - \mu_0)}{\sigma} < z_{\alpha}$$

2. One-Sided Hypothesis Test: Binomial Model with Unknown p using a Gaussian Approximation (Valid if $np_0(1-p_0) > 5$).

$H_0 : p = p_0$

A. $H_A : p > p_0$

Reject $H_0 : p = p_0$ with significance level α if

$$z = \frac{n^{\frac{1}{2}}(\hat{p} - p_0)}{[p_0(1-p_0)]^{\frac{1}{2}}} > z_{1-\alpha}$$

B. $H_A : p < p_0$

Reject $H_0 : p = p_0$ with significance level α if

$$z = \frac{n^{\frac{1}{2}}(\hat{p} - p_0)}{[p_0(1-p_0)]^{\frac{1}{2}}} < z_{\alpha}$$

3. Sample Size Formula Gaussian Mean, One-Sided Alternative: Type I Error = α; Power = $1-\beta$ and Effect = $\Delta = |\mu_0 - \mu_A|$.

$$n = \frac{\sigma^2 (z_{1-\beta} + z_{1-\alpha})^2}{\Delta^2}$$

4. Two-Sided Hypothesis Test: Gaussian Distribution Unknown Mean with Known Variance

$$H_0 : \mu = 0$$
$$H_A : \mu \neq 0$$

Reject $H_0 : \mu = \mu_0$ with significance level α if

$$|z| = \left| \frac{n^{\frac{1}{2}} (\bar{x} - \mu_0)}{\sigma} \right| > z_{1-\alpha/2}$$

5. One-Sided Hypothesis Test: Binomial Model with Unknown p using a Gaussian Approximation (valid if $np_0(1-p_0) > 5$).

$$H_0 : p = p_0$$

Reject $H_0 : p = p_0$ with significance level α if

$$|z| = \left| \frac{n^{\frac{1}{2}} (\hat{p} - p_0)}{[p_0(1-p_0)]^{\frac{1}{2}}} \right| > z_{1-\alpha}$$

6. Sample Size Formula Gaussian Mean, Two-Sided Alternative: Type I error = α; power = $1-\beta$ and effect = $\Delta = |\mu_0 - \mu_A|$.

$$n = \frac{\sigma^2}{\Delta^2} (z_{1-\alpha/2} + z_{1-\beta})^2.$$

7. Gaussian Distribution Known Unknown Mean with Unknown Variance: One-Sample t-Test.

$$H_0 : \mu = \mu_0$$
$$H_A : \mu \neq \mu_0$$

t-statistic

$$t = \frac{n^{\frac{1}{2}} (\bar{x} - \mu_0)}{s}$$

where

$$s^2 = (n-1)^{-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

Reject H_0 with significance level α if

$$|t| > t_{n-1,1-\alpha/2}$$

8. Binomial Exact Method ($np_0(1-p_0) < 5$).

$H_0 : p = p_0$

The p-value depends on whether $\hat{p} \le p_0$ or $\hat{p} > p_0$ where $\hat{p} = \frac{k}{n}$

A. If $\hat{p} \le p_0$

$$\frac{p-value}{2} = \Pr(\le k \text{ successes in } n \text{ trials} \mid H_0)$$
$$= \sum_{j=0}^{k} \binom{n}{j} p_0^j (1-p_0)^{n-j}$$

B. If $\hat{p} > p_0$

$$\frac{p-value}{2} = \Pr(\ge k \text{ successes in } n \text{ trials} \mid H_0)$$
$$= \sum_{j=k}^{n} \binom{n}{j} p_0^j (1-p_0)^{n-j}$$

Reject $H_0 : p = p_0$ with significance level α if $p-value < \alpha$.

9. $100 \times (1-\alpha)$ **Confidence Intervals Can Be Used To Perform Two-Sided Hypothesis Tests** with significance level α .

A. z-test (Gaussian mean and variance known)

$$\bar{x} \pm z_{1-\alpha/2} \frac{\sigma}{n^{\frac{1}{2}}}$$

B. z-test (Binomial proportion)

$$\hat{p} \pm z_{1-\alpha/2} \left[\frac{\hat{p}(1-\hat{p})}{n} \right]^{\frac{1}{2}}$$

C. t-test (Gaussian mean and variance unknown)

$$\bar{x} \pm t_{n-1,1-\alpha/2} \frac{s}{n^{\frac{1}{2}}}$$

10. Two Gaussian Samples with Unknown Means and Unknown but Equal Variances: Two-Sample t Test

$$H_0 : \mu_T = \mu_c$$

$$H_A : \mu_T \neq \mu_c$$

$$x_i^T \sim N(\mu_T, \sigma_T^2) \qquad i = 1, \ldots, n_T$$

$$x_j^c \sim N(\mu_c, \sigma_c^2) \qquad j = 1, \ldots, n_c$$

Assume $\sigma_T^2 = \sigma_c^2 = \sigma^2$. The unbiased estimate of σ^2 is:

$$s^2 = \frac{(n_T - 1)s_T^2 + (n_c - 1)s_c^2}{n_T + n_c - 2}$$

where

$$s_T^2 = (n_T - 1)^{-1} \sum_{j=1}^{n_T} (x_j^T - \bar{x}_T)^2$$

$$s_c^2 = (n_c - 1)^{-1} \sum_{i=1}^{n_c} (x_i^c - \bar{x}_c)^2.$$

Test statistic

$$t = \frac{\bar{x}_T - \bar{x}_c}{s(\frac{1}{n_T} + \frac{1}{n_c})^{\frac{1}{2}}}$$

Reject H_0 with significance level α if

$$|t| > t_{n, 1-\alpha/2}$$

where $n = n_T + n_c - 2.$

11. $100 \times (1 - \alpha)$ **Confidence Interval for the True Difference in the Means from Gaussian Distributions with Unknown but Equal Variances.**

$$\bar{x}_T - \bar{x}_c \pm t_{n, 1-\alpha/2} s \left(\frac{1}{n_T} + \frac{1}{n_C} \right)^{\frac{1}{2}}$$

12. Two Gaussian Samples with Unknown Means and Unknown and Unequal Variances: Two-Sample t Test.

$$H_0 : \mu_T = \mu_c$$

$$H_A : \mu_T \neq \mu_c$$

Test statistic (Satterthwaite Approximation)

$$t = \frac{\bar{x}_T - \bar{x}_c}{\left(\frac{s_T^2}{n_T} + \frac{s_c^2}{n_c}\right)^{\frac{1}{2}}}$$

where the number of degrees of freedom is

$$d' = \frac{(\frac{s_T^2}{n_T} + \frac{s_c^2}{n_c})^2}{\left(\frac{s_T^2}{n_T}\right)^2 (n_T - 1)^{-1} + \left(\frac{s_c^2}{n_c}\right)^2 (n_c - 1)^{-1}}$$

13. Two-Sample Test for Binomial Proportions (Gaussian Approximation: Valid if $n_1 \hat{p}_1 (1 - \hat{p}_1) > 5$ and $n_2 \hat{p}_2 (1 - \hat{p}_2) > 5$)

$$x_j^1 \sim B(n_1, p_1) \qquad j = 1, ..., n_1$$

$$x_j^2 \sim B(n_2, p_2) \qquad j = 1, ..., n_2$$

$$H_0 : p_1 = p_2$$

$$H_A : p_1 \neq p_2$$

z- statistic

$$z = \frac{|\hat{p}_1 - \hat{p}_2|}{[\hat{p}(1 - \hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)]^{\frac{1}{2}}}$$

where

$$\hat{p}_1 = \frac{k_1}{n_1}$$

$$\hat{p}_2 = \frac{k_2}{n_2}$$

$$\hat{p} = \frac{n_1 \hat{p}_1 + n_2 \hat{p}_2}{n_1 + n_2} = \frac{k_1 + k_2}{n_1 + n_2}$$

Reject H_0 with significance level α if

$$z > z_{1-\alpha/2}$$

p- value is $p = 2[1-\Phi(z)]$.

14. Two-Sample Test for Binomial Proportions (Gaussian Approximation: Valid if $n_1\hat{p}_1(1-\hat{p}_1) > 5$ and $n_2\hat{p}_2(1-\hat{p}_2) > 5$) with Continuity Correction.

$$H_0 : p_1 = p_2$$

$$H_A : p_1 \neq p_2$$

$$z = \frac{|\hat{p}_1 - \hat{p}_2| - \left(\frac{1}{2n_1} + \frac{1}{2n_2}\right)}{[\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)]^{\frac{1}{2}}}$$

Reject H_0 with significance level α if

$$z > z_{1-\alpha/2}.$$

15. $100\%(1-\alpha)$ Confidence Interval for the Difference of Two Binomial Proportions.

$$\hat{p}_1 - \hat{p}_2 \pm z_{1-\alpha/2}[\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)]^{\frac{1}{2}}$$

16. Two Sided Hypothesis Test for the Difference of Two Binomial Proportions. The $100\%(1-\alpha)$ confidence interval can be used to conduct a test of the null hypothesis $H_0 : p_1 = p_2$ versus the two-sided alternative $H_A : p_1 \neq p_2$ at significance level α. Reject $H_0 : p_1 = p_2$ if 0 is not in the $100\%(1-\alpha)$ confidence interval.