**Ekman layers, friction & geostrophic flow**

The homework problem (pucks_on_ice) illustrates several important points:

1. Particles move perpendicular to the applied force, to the right in the northern hemisphere, to the left in the southern hemisphere
2. Particles do not move steadily but oscillate with an inertial frequency, equal to $2\Omega\sin(\theta)$, where θ is the latitude
3. After the external force is removed, particle motion decays except when close to the "bump" where they continue to rotate around the bump with high elevation on the right (left) in the northern (southern) hemisphere
4. Due to friction, particles near the bump slowly slide down the slope

Point 1 is an illustration of Ekman layers in the ocean. A force (wind stress) pushes them at right angles in the rotating system. One might wonder how it is possible in such a "world" to drive from California to Seattle: one would have to head westward in your car (towards the Pacific Ocean) to get there! But in fact, friction for automobiles is rather large: the tires constantly impeded motion except in the direction the car is pointed. So don't try this experiment!
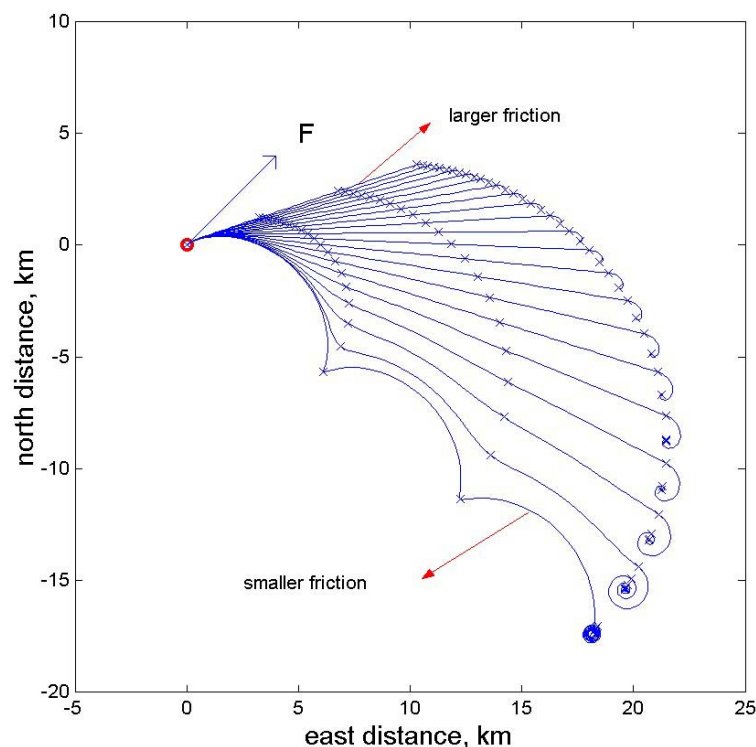
The variation of the particle motion with ever-increasing friction is shown in the figure above. As friction is increased (relative to f) the particle moves more in the direction of the applied force (blue arrow). The oscillation amplitude also decreases. In the limit in which friction is much larger than f, the motion is in the direction of the applied force.

The dynamics behind the "Pucks on Ice" problem is all contained in the equation for the change in momentum of a particle of mass "m" moving in a rotating system about a frozen, bumpy surface "h(x,y)" under the action of a body force, "F" and weak friction "r" . As we have seen, when friction is weak enough (in one limit discussed above), the pucks can do some things that are initially surprising when the body force is switched on and then off after a short time (3 days in the problem). The dynamical balance is given as

$$m[\frac{d\vec{v}}{dt} + 2\vec{\Omega} \times \vec{v}] = -mr\vec{v} - mg\nabla h + \vec{F}$$

$$(1) \quad (2) \quad (3) \quad (4) \quad (5)$$

Each of the terms has been numbered to facilitate some discussion. One feature of the solution is that shortly after the body force is either switched on or off, particles undergo oscillations in a circle. This oscillation is an inertial oscillation and is characterized as a balance between terms (1) & (2).
Inertial oscillations: terms (1) & (2)

For those particles near the bumpy topography, their motion will be such as to move with velocities that have the high topography on their right (in the Northern Hemisphere, left in the S. Hemisphere). This balance is called the geostrophic balance.
Geostrophic flow: terms (2) & (4)

Under the action of the body force, particles will move at right angles to the force, to the right in the N. hemisphere (left in S. Hemisphere). This is represents a balance between terms (2) and (5) and is the fundamental balance in the wind-driven, Ekman transport (after V. Walfrid Ekman).
Ekman flow: terms (2) & (5) [&(3)]

Weak friction will modify all of the above, and in the problem, the above balances are never exactly followed, only approximately, in different regions

and at different times. Yet these "balances" lie at the core of wind-driven motion in the ocean. In all of these cases, the Coriolis force (2) is central.

Ekman flow in the ocean

Now consider a non-frozen ocean forced by an applied wind stress in the direction of the blue vector. For the surface layer, turbulence is large and the effective friction is large: so a particle will move downwind but slightly to the right (in the N. hemisphere, which we will assume in what follows). The layer below, which is causing the drag on the surface layer then wants to move too. It will move to the right of the overlying surface layer for the same reasons. This continues in the vertical until the turbulence, which couples the "layers" in the water column, eventually dies away and there is no subsequent motion. The resulting velocity vector looks like a spiral: going from slightly downwind but to the right of the wind, rotating to the clockwise and decaying with increasing depth. This was first pointed out by Ekman with a very simple model of the frictional coupling in 1905, and convincingly demonstrated in the ocean by Price, Weller and Schudlich (Science, vol. 238, pp 1534-1538) in 1987. The net transport in this frictional layer is almost exactly to the right of the stress as predicted! We call this surface layer of wind stress influence the Ekman layer. In the figure below, we can see this almost exact balance of net transport integrated over the surface Ekman layer, which is about 25m depth, and the wind stress, such that the net transport is the right of the wind stress (in the N. Hemisphere).

Image removed due to copyright concerns.

Frictional or Ekman layers can exist near the ocean bottom as well. This was also illustrated in the homework problem as the pucks slowly slid down the sloping topography because of friction. The main motion of the pucks was clockwise around the topographic "bump". This reflects the **geostrophic balance** between motion and the slope in surface elevation. Since frictional coupling is again strong when there is flow in near contact with the ocean bottom, this ageostrophic (non-geostrophic) motion, again deflects the trajectories of particles near to the bottom to the *left* of those further away from the bottom. This creates a spiral in the opposite direction with net frictional flow to the left of the overlying fluid (in the N. hemisphere). A similar result is obtained for geostrophic flow around a "dimple". Near the bottom, friction upsets this balance and there is again a flow down the pressure gradient and a turning of the flow to the *left* of the interior motion.

Notes on the Geostrophic Balance

Consider the diagram at the right. The ocean consists of two density layers of uniform density ρ_1 & ρ_2. The height of the free surface is given by $h(x)$, that of the interface by – $H(x)$ and of the flat lower surface $-H_0$: it needn't be the ocean bottom, only a level surface. We will use the hydrostatic relation to determine the pressure on the lower surface. It is given by the following:

Z=h
Z=0
ρ_1
ρ_2
Z= -H
Z= -H$_0$
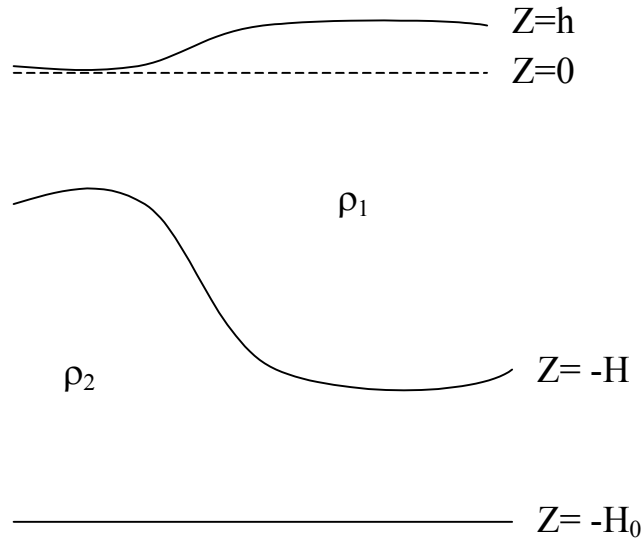
$$P(z = -H_0) = g\rho_1(h + H) + g\rho_2(H_0 - H)$$

If at this surface there is no horizontal pressure gradient, $P_x = 0$ and we get the following (after canceling out g, which appears in both terms):

$$P_x = 0 \rightarrow 0 = \rho_1(h_x + H_x) - \rho_2 H_x, \text{ or}$$

$$h_x = \frac{\Delta\rho}{\rho_1} H_x, \text{ where } \Delta\rho \equiv \rho_2 - \rho_1$$

Since $\Delta\rho/\rho_1 \ll 1$, the interface thickens and deepens much more than the free surface rises. For the Gulf Stream, the above picture is what one would see standing near Cape Hatteras and looking to the east. The surface rises about 1 meter across the Gulf Stream due to the strong flow at the surface, but this flow decays rapidly with depth. The density contrast across the pycnocline makes $\Delta\rho/\rho_1 \sim 2\text{x}10^{-3}$ so the pycnocline slopes down to the south (right) about 500m across the Gulf Stream. Of course, the above situation is only an idealization of the actual situation (the class will have a homework problem using real data), but this illustrates the degree to which the density field can compensate for a non-level free surface expression.

We will now look at the equations expressing the geostrophic balance for baroclinic motion. In our grand equations, we can write the balance for geostrophic motion [$(u,v)=(u_g,v_g)$] as

$$-\rho f v_g = -\left[\frac{\partial p}{\partial x}\right]_\Phi$$

$$\rho f u_g = -\left[\frac{\partial p}{\partial y}\right]_\Phi$$

where we have noted that the pressure gradients in (x,y) are calculated on a constant geopotential surface for clarity. Now recall from the definition of geopotential surfaces that there is no change in potential energy as a particle is moved in a surface of constant Φ, and that $\Phi(p(x,y),x,y)$ explicitly depends on pressure as well as horizontal position. We can use this to make a change of variables in the above. For example, if

$$\Phi = \Phi(p(x,y),x,y), then$$

$$\delta\Phi = \left[\frac{\partial\Phi}{\partial p}\right]_{x,y}\delta p + \left[\frac{\partial\Phi}{\partial x}\right]_{p,y}\delta x + \left[\frac{\partial\Phi}{\partial y}\right]_{p,x}\delta y.$$

$$For\ \delta\Phi, \delta y = 0,$$

$$\left[\frac{\partial p}{\partial x}\right]_\Phi \equiv \left[\frac{\delta p}{\delta x}\right]_{\Phi,y} = -\left[\frac{\partial\Phi}{\partial x}\right]_{p,y} \bigg/ \left[\frac{\partial\Phi}{\partial p}\right]_{x,y}.$$

So we can write

$$\left[\frac{\partial p}{\partial x}\right]_\Phi = -\left[\frac{\partial\Phi}{\partial x}\right]_p \bigg/ \left[\frac{\partial\Phi}{\partial p}\right]_{x,y} = \rho\left[\frac{\partial\Phi}{\partial x}\right]_p$$

$$\left[\frac{\partial p}{\partial y}\right]_\Phi = -\left[\frac{\partial\Phi}{\partial y}\right]_p \bigg/ \left[\frac{\partial\Phi}{\partial p}\right]_{x,y} = \rho\left[\frac{\partial\Phi}{\partial y}\right]_p.$$

And our geostrophic balance becomes

$$-fv_g(p) = -\left[\frac{\partial \Phi}{\partial x}\right]_p = -\frac{\partial \Phi(0)}{\partial x} + \frac{\partial \Delta D(p)}{\partial x}$$

$$fu_g(p) = -\left[\frac{\partial \Phi}{\partial y}\right]_p = -\frac{\partial \Phi(0)}{\partial y} + \frac{\partial \Delta D(p)}{\partial y}$$

where we have used the definition of geopotential presented earlier in terms of dynamic height and also made use of the fact that there is no *horizontal* variation of one of the terms in the definition of geopotential which is proportional to the reference specific volume anomaly α_0. The last pair of equations can be further simplified recognizing that for $p = 0$, $\Delta D = 0$, thus

$$-f[v_g(0) - v_g(p)] = -\frac{\partial \Delta D(p)}{\partial x}$$

$$f[u_g(0) - u_g(p)] = -\frac{\partial \Delta D(p)}{\partial y}$$

This is the form of the geostrophic equations most commonly used since the gradient of geopotential at the free surface is not easily "measured". It expresses how velocity will change with pressure based on horizontal gradients of dynamic height. Without any other information, on cannot determine the geostrophic velocity at a point, only its variation with pressure (or depth): there is an unknown constant, which can be determined by fiat: [saying that the velocity at some reference pressure must be zero] or by other means [such as equating the velocity at some depth with direct measurements of velocity and then requiring that measured velocity to be "geostrophic"]. [reference for some of this is Fofonoff in *The Sea*, Vol. 1]. In a later homework problem, we will calculate geostrophic currents using actual oceanographic data. Use of the above pair of equations will be *essential* as well as understanding how to use the concept of a "deep reference pressure" as a level of no motion.

On an f-plane, contours of constant dynamic height are equivalent to streamlines of the geostrophic flow relative to some assumed "level of no motion". While this is a convenient fact and of much use in examining spatial maps of dynamic height from hydrographic data, one must always be aware of the limitations of this approximation. Because f can vary with latitude, this approximation breaks down for whole ocean basins.