Q1. Third order, direct space time method. i) Derive a third order accurate (time and space) finite difference approximation to the linear advection problem

$$\partial_t \theta + c\partial_x \theta = 0 \tag{1}$$

where $c > 0$ a positive constant flow. The resulting scheme should take the form

$$\frac{1}{\Delta t}(\theta_i^{n+1} - \theta_i^n) = -\frac{c}{\Delta x}(\delta\theta_{i-2}^n + \gamma\theta_{i-1}^n + \beta\theta_i^n + \alpha\theta_{i+1}^n) \tag{2}$$

where α, β, γ and δ are factors that you will determine. Assume a regular grid with index i such that $x_i = i\Delta x$ and $\theta_i = \theta(x_i)$. Hint: You will need higher time derivatives of the above governing equation to eliminate the first and second order time truncation terms.

Answer: Substituting in the Taylor expansion for each of θ_i^{n+1}, θ_{i-2}^n, θ_{i+1}^n and θ_{i+1}^n the higher order derivatives $\partial_{tt}\theta = c^2\partial_{xx}\theta$ and $\partial_{ttt}\theta = -c^3\partial_{xxx}\theta$ we get

$$\begin{aligned}
&\frac{1}{\Delta t}(\theta_i^{n+1} - \theta_i^n) + \frac{c}{\Delta x}(\delta\theta_{i-2}^n + \gamma\theta_{i-1}^n + \beta\theta_i^n + \alpha\theta_{i+1}^n) \\
= \;& \partial_t\theta + \frac{\Delta t}{2}\partial_{tt}\theta + \frac{\Delta t^2}{3!}\partial_{ttt}\theta + \frac{c}{\Delta x}(\delta + \gamma + \beta + \alpha)\theta_i + c(-2\delta - \gamma + \alpha)\partial_x\theta \\
& + \frac{c\Delta x}{2}(4\delta + \gamma + \alpha)\partial_{xx}\theta + \frac{c\Delta x^2}{3!}(-8\delta - \gamma + \alpha)\partial_{xxx}\theta + O(\Delta t^3, \Delta x^3) \\
= \;& \partial_t\theta + c(-2\delta - \gamma + \alpha)\partial_x\theta \\
& + \frac{c}{\Delta x}(\delta + \gamma + \beta + \alpha)\theta_i \\
& + \frac{c^2\Delta t}{2}\partial_{xx}\theta + \frac{c\Delta x}{2}(4\delta + \gamma + \alpha)\partial_{xx}\theta \\
& - \frac{c^3\Delta t^2}{3!}\partial_{xxx}\theta + \frac{c\Delta x^2}{3!}(-8\delta - \gamma + \alpha)\partial_{xxx}\theta + O(\Delta t^3, \Delta x^3)
\end{aligned}$$

Eliminating all terms that do not appear in the governing equation we find

$$\begin{aligned}
\delta + \gamma + \beta + \alpha &= 0 \\
-2\delta - \gamma + \alpha &= 1 \\
4\delta + \gamma + \alpha &= -C \\
-8\delta - \gamma + \alpha &= C^2
\end{aligned}$$

where $C = \frac{c\Delta t}{\Delta x}$.

$$\beta = -\alpha - \gamma - \delta$$

$$\begin{aligned}
\gamma &= -1+\alpha-2\delta \\
2\delta+2\alpha &= 1-C \\
-4\delta+2\alpha &= -C(1-C)
\end{aligned}$$

Solving for α, β, γ and δ

$$\begin{aligned}
6\delta &= (1+C)(1-C) = 1-C^2 \\
6\alpha &= (2-C)(1-C) = 2-3C+C^2 \\
6\gamma &= -6-3C(1-C) = -6-3C+3C^2 \\
6\beta &= 3+6C-3C^2
\end{aligned}$$

ii) Derive the discrete flux F that when used in the difference equation

$$\frac{1}{\Delta t}(\theta_i^{n+1}-\theta_i^n) = -\frac{1}{\Delta x}(F_{i+\frac{1}{2}}-F_{i-\frac{1}{2}}) \tag{3}$$

makes it equivalent to the difference equation (1). Hint: F takes the form

$$F_{i+\frac{1}{2}} = c[\theta_i + d_1(\theta_i-\theta_{i-1}) + d_0(\theta_{i+1}-\theta_i)] \tag{4}$$

where d_0 and d_1 are functions of the Courant number, $C = \frac{c\Delta t}{\Delta x}$.

Answer: Given that the flux takes the form or (4) we can write out (3) in terms of θ alone:

$$\begin{aligned}
\frac{1}{\Delta t}(\theta_i^{n+1}-\theta_i^n) &= -\frac{c}{\Delta x}\Big[\theta_i^n + d_1(\theta_i^n-\theta_{i-1}^n) + d_0(\theta_{i+1}^n-\theta_i^n) \\
&\qquad -\theta_{i-1}^n - d_1(\theta_{i-1}^n-\theta_{i-2}^n) - d_0(\theta_i^n-\theta_{i-1}^n)\Big] \\
&= -\frac{c}{\Delta x}[d_1\theta_{i-2}^n + (-1+d_0-2d_1)\theta_{i-1}^n \\
&\qquad +(1-2d_0+d_1)\theta_i^n + d_0\theta_{i+1}^n] \\
&= -\frac{c}{\Delta x}[\delta\theta_{i-2}^n + \gamma\theta_{i-1}^n + \beta\theta_i^n + \alpha\theta_{i+1}^n]
\end{aligned}$$

Equating coefficients gives:

$$\begin{aligned}
d_0 &= \alpha = \frac{1}{6}(2-C)(1-C) \\
d_1 &= \delta = \frac{1}{6}(1+C)(1-C)
\end{aligned}$$

Matching the other two coefficients supply a sanity check:

$$
\begin{aligned}
-1 + d_0 - 2d_1 &= -1 + \alpha - 2\delta = \gamma \\
1 - 2d_0 + d_1 &= 1 - 2\alpha + \delta = \beta
\end{aligned}
$$

iii) Consider this flux in the limit of vanishing Courant number. What discretization does this correspond to (see your previous problem set)?

Answer: In the limit of $C \to 0$, $d_0 \to \frac{1}{3}$ and $d_1 \to \frac{1}{6}$. This looks like the third order finite difference flux obtained by considering only the spatial truncation errors (as in problem set 1).

Q2. Finite volume method Again, consider the linear advection problem cast in flux form (3) where $F = c\theta$ with $c > 0$ on a regular grid. We will consider the flux of properties across the point $x = x_{i+\frac{1}{2}}$ as the average of the upstream time-average of

$$F_{i+\frac{1}{2}} = \frac{1}{\Delta t} \int_{x_{i+\frac{1}{2}} - c\Delta t}^{x_{i+\frac{1}{2}}} \theta(x)\, dx \tag{5}$$

i) Consider the distribution of θ at time $t = n\Delta t$ assuming that θ is piecewise constant in the finite volume Δx around each point x_i (i.e. θ is constant with value θ_i between $x_i - \frac{1}{2}\Delta x$ and $x_i + \frac{1}{2}\Delta x$.).

a) Evaluate $F_{i+\frac{1}{2}}$ in equation (5). You may assume that $\Delta t \leq \Delta x / c$.

Answer:

$$F_{i+\frac{1}{2}} = \frac{1}{\Delta t} \int_{x_{i+\frac{1}{2}} - c\Delta t}^{x_{i+\frac{1}{2}}} \theta_i\, dx = \frac{1}{\Delta t} [\theta_i x]_{-c\Delta t}^{0} = c\theta_i$$

b) What is this scheme usually called?

Answer: It is the F.T.U.S. scheme.

c) To make this calculation, why is it useful to assume $\Delta t \leq \Delta x / c$?

Answer: Because the value of θ is discontinuous at a distance $c\Delta t$ to the left of $x_{i+\frac{1}{2}}$.

d) Now re-evaluate $F_{i+\frac{1}{2}}$ in equation (5), this time assuming $\Delta x / c \leq \Delta t \leq 2\Delta x / c$.

Answer:

$$F_{i+\frac{1}{2}} = \frac{1}{\Delta t} \int_{x_{i+\frac{1}{2}} - c\Delta t}^{x_{i+\frac{1}{2}}} \theta(x)\, dx$$

$$
\begin{aligned}
&= \frac{1}{\Delta t}\int_{x_{i+\frac{1}{2}}-\Delta x-c'\Delta t}^{x_{i+\frac{1}{2}}} \theta(x)\,dx \\
&= \frac{1}{\Delta t}\int_{x_{i+\frac{1}{2}}-\Delta x}^{x_{i+\frac{1}{2}}} \theta_i\,dx + \frac{1}{\Delta t}\int_{x_{i+\frac{1}{2}}-\Delta x-c'\Delta t}^{x_{i+\frac{1}{2}}-\Delta x} \theta_{i-1}\,dx \\
&= \frac{1}{\Delta t}[\theta_i x]_{-\Delta x}^{0} + \frac{1}{\Delta t}[\theta_{i-1}x]_{-c'\Delta t}^{0} \\
&= \frac{\Delta x}{\Delta t}\theta_i + c'\theta_{i-1} = \frac{\Delta x}{\Delta t}\theta_i + (c - \frac{\Delta x}{\Delta t})\theta_{i-1}
\end{aligned}
$$

e) Generalize you answers for (a) and (d) so that you can evaluate $F_{i+\frac{1}{2}}$ using one expression assuming $\Delta t \leq 2\Delta x/c$. Hint: you will need to use the min and max functions:

$$
\begin{aligned}
min(a,b) &= \begin{cases} a & \text{if} \quad a \leq b \\ b & \text{if} \quad a > b \end{cases} \\
max(a,b) &= \begin{cases} a & \text{if} \quad a \geq b \\ b & \text{if} \quad a < b \end{cases}
\end{aligned}
$$

Answer:

$$
F_{i+\frac{1}{2}} = \min{(c, \frac{\Delta x}{\Delta t})}\theta_i + \left(\max{(c, \frac{\Delta x}{\Delta t})} - \frac{\Delta x}{\Delta t}\right)\theta_{i-1}
$$

ii) Consider the distribution of θ at time $t = n\Delta t$ to be piecewise linear between the nodes x_i.

a) Write down θ as a function of x in the interval $x_i \leq x \leq x_{i+1}$. Hint: this is simply linear interpolation between the values θ_i and θ_{i+1}.

Answer:

$$
\begin{aligned}
\theta(x) &= \frac{(x_{i+1}-x)\theta_i + (x-x_i)\theta_{i+1}}{\Delta x} \\
&= \frac{(\frac{\Delta x}{2} - x')\theta_i + (\frac{\Delta x}{2} + x')\theta_{i+1}}{\Delta x} \quad \textit{where} \quad x' = x - x_{i+\frac{1}{2}}
\end{aligned}
$$

b) Evaluate $F_{i+\frac{1}{2}}$ in equation (5) assuming a piecewise linear distribution. You may assume that $\Delta t \leq \frac{1}{2}\Delta x/c$.

Answer:

$$
\begin{aligned}
F_{i+\frac{1}{2}} &= \frac{1}{\Delta t}\int_{x_{i+\frac{1}{2}}-c\Delta t}^{x_{i+\frac{1}{2}}} \theta(x)\,dx \\
&= \frac{1}{\Delta t}\int_{-c\Delta t}^{0} \theta(x')\,dx' \quad \textit{where} \quad x' = x - x_{i+\frac{1}{2}} \\
&= \frac{1}{\Delta t}\int_{-c\Delta t}^{0} \frac{1}{2}(\theta_i + \theta_{i+1})\,dx' + \frac{1}{\Delta t}\int_{-c\Delta t}^{0} \frac{x'}{\Delta x}(\theta_{i+1} - \theta_i)\,dx' \\
&= \frac{1}{\Delta t}[\frac{x'}{2}(\theta_i + \theta_{i+1})]_{-c\Delta t}^{0} + \frac{1}{\Delta t}[\frac{x'^2}{2\Delta x}(\theta_{i+1} - \theta_i)]_{-c\Delta t}^{0} \\
&= \frac{c}{2}(\theta_i + \theta_{i+1}) - \frac{c^2\Delta t}{2\Delta x}(\theta_{i+1} - \theta_i)
\end{aligned}
$$

c) What is this scheme usually called?

Answer: It is the Lax-Wendroff scheme.

iii) Consider the distribution of θ at time $t = n\Delta t$ to be piecewise quadratic between the nodes x_i.

a) Write down θ as a function of x in the interval $x_i \leq x \leq x_{i+1}$ by fitting a quadratic function to the nodes θ_{i-1}, θ_i and θ_{i+1} (i.e $\theta(x_j) = \theta_j$ at $j = i-1, i, i+1$).

Answer: Assume

$$
\theta(x) = \alpha + 2\beta\frac{(x - x_{i+\frac{1}{2}})}{\Delta x} + 3\gamma\frac{(x - x_{i+\frac{1}{2}})^2}{\Delta x^2}
$$

then

$$
\begin{aligned}
\alpha - 3\beta + \frac{27}{4}\gamma &= \theta_{i-1} \\
\alpha - \beta + \frac{3}{4}\gamma &= \theta_i \\
\alpha + \beta + \frac{3}{4}\gamma &= \theta_{i+1}
\end{aligned}
$$

or

$$
\begin{aligned}
\alpha + \frac{3}{4}\gamma &= \frac{1}{2}\theta_i + \frac{1}{2}\theta_{i+1} \\
\alpha + \frac{9}{4}\gamma &= \frac{1}{4}\theta_{i-1} + \frac{3}{4}\theta_{i+1} \\
2\beta &= \theta_{i+1} - \theta_i
\end{aligned}
$$

Solving for α, β and γ:

$$\begin{aligned}
\gamma &= \frac{1}{6}\theta_{i-1} - \frac{2}{6}\theta_i + \frac{1}{6}\theta_{i+1} \\
\beta &= -\frac{1}{2}\theta_i + \frac{1}{2}\theta_{i+1} \\
\alpha &= -\frac{1}{8}\theta_{i-1} + \frac{6}{8}\theta_i + \frac{3}{8}\theta_{i+1}
\end{aligned}$$

b) Evaluate $F_{i+\frac{1}{2}}$ in equation (5) assuming a piecewise quadratic distribution.

Answer:

$$\begin{aligned}
F_{i+\frac{1}{2}} &= \frac{1}{\Delta t}\int_{-c\Delta t}^{0} \theta(x')\,dx' \\
&= \frac{1}{\Delta t}\left[\alpha x + \beta\frac{x^2}{\Delta x} + \gamma\frac{x^3}{\Delta x^2}\right]_{-c\Delta t}^{0} \\
&= c\left(\alpha - \frac{c\Delta t}{\Delta x}\beta + \frac{c^2\Delta t^2}{\Delta x^2}\gamma\right) \\
&= c\left[(-\frac{1}{8} + \frac{c^2\Delta t^2}{6\Delta x^2})\theta_{i-1} + (\frac{6}{8} - \frac{c\Delta t}{2\Delta x} + \frac{2c^2\Delta t^2}{6\Delta x^2})\theta_i \right. \\
&\qquad \left. +(\frac{3}{8} + \frac{c\Delta t}{2\Delta x} - \frac{c^2\Delta t^2}{6\Delta x^2})\theta_{i+1}\right]
\end{aligned}$$

c) In the limit of vanishing time-step, what scheme does the flux in (b) approach?

Answer:

$$F_{i+\frac{1}{2}} = -\frac{1}{8}\theta_{i-1} + \frac{6}{8}\theta_i + \frac{3}{8}\theta_{i+1}$$

which is the third-order form of the interpolated flux but is not third order for the advection equation (see Problem Set 1).

iv) Again, consider the distribution of θ at time $t = n\Delta t$ to be piecewise quadratic in the interval $x_i \le x \le x_{i+1}$ and to take the form:

$$\theta(x) = \alpha + 2\beta\frac{(x - x_{i+\frac{1}{2}})}{\Delta x} + 3\gamma\frac{(x - x_{i+\frac{1}{2}})^2}{\Delta x^2}. \qquad (6)$$

a) Find α, β and γ so that the spatial average over each finite volume (Δx) around x_{i-1}, x_i and x_{i+1} equals θ_{i-1}, θ_i and θ_{i+1} respectively. Note that this is different to fitting the quadratic function at the nodes as you did in part (iii).

Answer: Using $x' = x - x_{i+\frac{1}{2}}$

$$
\begin{aligned}
\theta(x) &= \alpha + 2\beta\frac{x'}{\Delta x} + 3\gamma\frac{x'^2}{\Delta x^2} \\
\Delta x\theta_{i+1} &= \left[\alpha x' + \beta\frac{x'^2}{\Delta x} + \gamma\frac{x'^3}{\Delta x^2}\right]_0^{\Delta x} \\
\Delta x\theta_{i} &= \left[\alpha x' + \beta\frac{x'^2}{\Delta x} + \gamma\frac{x'^3}{\Delta x^2}\right]_{-\Delta x}^{0} \\
\Delta x\theta_{i-1} &= \left[\alpha x' + \beta\frac{x'^2}{\Delta x} + \gamma\frac{x'^3}{\Delta x^2}\right]_{-2\Delta x}^{-\Delta x}
\end{aligned}
$$

or

$$
\begin{aligned}
\theta_{i+1} &= \alpha + \beta + \gamma \\
\theta_{i} &= \alpha - \beta + \gamma \\
\theta_{i-1} &= \alpha - 3\beta + 7\gamma
\end{aligned}
$$

Solution:

$$
\begin{aligned}
\gamma &= \frac{1}{6}\theta_{i-1} - \frac{2}{6}\theta_i + \frac{1}{6}\theta_{i+1} \\
\beta &= -\frac{1}{2}\theta_i + \frac{1}{2}\theta_{i+1} \\
\alpha &= -\frac{1}{6}\theta_{i-1} + \frac{5}{6}\theta_i + \frac{2}{6}\theta_{i+1}
\end{aligned}
$$

b) Evaluate $F_{i+\frac{1}{2}}$ in equation (5) using the "finite volume" representation from (a).

Answer:

$$
\begin{aligned}
F_{i+\frac{1}{2}} &= \frac{1}{\Delta t}\int_{-c\Delta t}^{0} \theta(x')\,dx' \\
&= c\left(\alpha - \frac{c\Delta t}{\Delta x}\beta + \frac{c^2\Delta t^2}{\Delta x^2}\gamma\right) \\
&= c\left[-\frac{1}{6}(1 - \frac{c^2\Delta t^2}{6\Delta x^2})\theta_{i-1} + (\frac{5}{6} - \frac{c\Delta t}{2\Delta x} + \frac{2c^2\Delta t^2}{6\Delta x^2})\theta_i \right. \\
&\qquad \left. +(\frac{2}{6} + \frac{c\Delta t}{2\Delta x} - \frac{c^2\Delta t^2}{6\Delta x^2})\theta_{i+1}\right]
\end{aligned}
$$

c) What is this scheme usually called?

Answer: It is the 3rd order Direct-Space-Time scheme from Q1.

Q3. Discrete conservation of variance The average and difference operators are

$$\begin{aligned} \overline{\theta}^i &= \frac{1}{2}\left(\theta_{i+\frac{1}{2}} + \theta_{i-\frac{1}{2}}\right) \\ \delta_i\theta &= \theta_{i+\frac{1}{2}} - \theta_{i-\frac{1}{2}} \end{aligned}$$

a) Prove the discrete product rule

$$\delta_i(\overline{\theta}^i U) = \overline{U\delta_i\theta}^i + \theta\delta_i U.$$

Answer:

$$\begin{aligned} \delta_i(\overline{\theta}^i U) &= U_{i+\frac{1}{2}}\frac{1}{2}(\theta_{i+1} + \theta_i) - U_{i-\frac{1}{2}}\frac{1}{2}(\theta_i + \theta_{i-1}) \\ &= \frac{1}{2}U_{i+\frac{1}{2}}(\theta_{i+1} - \theta_i) + \frac{1}{2}U_{i-\frac{1}{2}}(\theta_i - \theta_{i-1}) + \theta_i(U_{i+\frac{1}{2}} - U_{i-\frac{1}{2}}) \\ &= \overline{U\delta_i\theta}^i + \theta\delta_i U \end{aligned}$$

b) Prove the discrete product rule

$$\delta_i(\theta\phi) = \overline{\theta}^i\delta_i\phi + \overline{\phi}^i\delta_i\theta.$$

Answer:

$$\begin{aligned} \delta_i(\theta\phi) &= \theta_{i+\frac{1}{2}}\phi_{i+\frac{1}{2}} - \theta_{i-\frac{1}{2}}\phi_{i-\frac{1}{2}} \\ &= \frac{1}{2}\theta_{i+\frac{1}{2}}\phi_{i+\frac{1}{2}} - \frac{1}{2}\theta_{i-\frac{1}{2}}\phi_{i-\frac{1}{2}} + \frac{1}{2}\theta_{i+\frac{1}{2}}\phi_{i+\frac{1}{2}} - \frac{1}{2}\theta_{i-\frac{1}{2}}\phi_{i-\frac{1}{2}} \\ &= \frac{1}{2}\theta_{i+\frac{1}{2}}\phi_{i+\frac{1}{2}} - \frac{1}{2}\theta_{i-\frac{1}{2}}\phi_{i-\frac{1}{2}} + \frac{1}{2}\theta_{i+\frac{1}{2}}\phi_{i+\frac{1}{2}} - \frac{1}{2}\theta_{i-\frac{1}{2}}\phi_{i-\frac{1}{2}} \\ &\quad + \frac{1}{2}\theta_{i+\frac{1}{2}}\phi_{i-\frac{1}{2}} + \frac{1}{2}\theta_{i-\frac{1}{2}}\phi_{i+\frac{1}{2}} - \frac{1}{2}\theta_{i+\frac{1}{2}}\phi_{i-\frac{1}{2}} - \frac{1}{2}\theta_{i-\frac{1}{2}}\phi_{i+\frac{1}{2}} \\ &= \frac{1}{2}(\theta_{i+\frac{1}{2}} + \theta_{i-\frac{1}{2}})(\phi_{i+\frac{1}{2}} - \phi_{i-\frac{1}{2}}) + \frac{1}{2}(\phi_{i+\frac{1}{2}} + \phi_{i-\frac{1}{2}})(\theta_{i+\frac{1}{2}} - \theta_{i-\frac{1}{2}}) \\ &= \overline{\theta}^i\delta_i\phi + \overline{\phi}^i\delta_i\theta \end{aligned}$$

c) A scalar advection equation and continuity equation are discretized

$$\begin{aligned}
\Delta x \Delta y \partial_t \theta + \delta_i(\overline{\theta}^i U \Delta y) + \delta_j(\overline{\theta}^j V \Delta x) &= 0 \\
\delta_i(U \Delta y) + \delta_j(V \Delta x) &= 0.
\end{aligned}$$

Prove that the global integral of variance ($\int\int \theta^2 \, dx \, dy$) is conserved given no normal flow at domain boundaries. Assume perfect treatment of the time derivative.

Answer:

$$\begin{aligned}
\theta \delta_i(\overline{\theta}^i U \Delta y) &= \delta_i(\overline{\theta}^{i^2} U \Delta y) - \overline{\overline{\theta}^i U \Delta y \delta_i \theta}^i \\
&= \delta_i(\overline{\theta}^{i^2} U \Delta y) - \overline{U \Delta y \frac{1}{2} \delta_i \theta^2}^i \\
&= \delta_i(\overline{\theta}^{i^2} U \Delta y) - \frac{1}{2}\delta_i(\overline{\theta^2}^i U \Delta y) + \frac{1}{2}\theta^2 \delta_i(U \Delta y) \\
&= \delta_i((\overline{\theta}^{i^2} - \frac{1}{2}\overline{\theta^2}^i) U \Delta y) + \frac{1}{2}\theta^2 \delta_i(U \Delta y)
\end{aligned}$$

Similarly

$$\theta \delta_j(\overline{\theta}^j V \Delta x) = \delta_j((\overline{\theta}^{j^2} - \frac{1}{2}\overline{\theta^2}^j) V \Delta x) + \frac{1}{2}\theta^2 \delta_j(V \Delta x)$$

Substituting in the the variance equation

$$\begin{aligned}
\frac{1}{2}\Delta x \Delta y \partial_t \theta^2 &= -\theta \delta_i(\overline{\theta}^i U \Delta y) - \theta \delta_j(\overline{\theta}^j V \Delta x) \\
&= -\delta_i((\overline{\theta}^{i^2} - \frac{1}{2}\overline{\theta^2}^i) U \Delta y) - \delta_j((\overline{\theta}^{j^2} - \frac{1}{2}\overline{\theta^2}^j) V \Delta x) \\
&\quad -\frac{1}{2}\left(\theta^2 \delta_i(U \Delta y) + \theta^2 \delta_j(V \Delta x)\right) \\
&= -\delta_i((\overline{\theta}^{i^2} - \frac{1}{2}\overline{\theta^2}^i) U \Delta y) - \delta_j((\overline{\theta}^{j^2} - \frac{1}{2}\overline{\theta^2}^j) V \Delta x) \\
&= \delta_i F^x + \delta_j F^y
\end{aligned}$$

Because the variance equation can be written in flux form and the involved fluxes vanish on the domain boundaries, the domain integrated variance is conserved.

Q4. Burgers equation (Matlab) Burgers equation is

$$\partial_t u + u\partial_x u \quad = \quad 0.$$

We will consider this equation in the re-entrant (periodic) domain $0 \leq x \leq 1$ (i.e. $u(x=1,t) = u(x=0,t)$ for all t).
i) Show that the continuous Burgers equation (globally) conserves $\int u^p \, dx$ where p is an integer.

Answer: Since

$$\partial_t u^p = pu^{(p-1)}\partial_t u$$

and

$$\partial_x u^{p+1} = (p+1)u^p \partial_x u$$

then

$$u^{(p-1)}(\partial_t u + u\partial_x u) = \frac{1}{p}\partial_t u^p + \frac{1}{p+1}\partial_x u^{p+1}$$

thus

$$\partial_t \int u^p \, dx = \int \partial_t u^p \, dx \quad = \quad \frac{-p}{p+1}\left[u^{p+1}\right] = 0$$

ii) a) Spatially discretize Burgers equation using centered second order difference but keeping a continuous time derivative. This is known as a differential-difference equation.

Answer:

$$\partial_t u_i \quad = \quad -u_i \frac{1}{2\Delta x}\left(u_{i+1} - u_{i-1}\right)$$

b) Show that although the differential-difference equation (ii.a) was not written as the divergence of a flux, that this form does conserve $< u >$ (volume mean of u) and that it can be equivilently written in the flux form

$$\partial_t u = -\frac{1}{\Delta x}\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right)$$

where $F_{i+\frac{1}{2}}$ takes a particular form.

Answer:

$$-u_i \frac{1}{2\Delta x}\left(u_{i+1} - u_{i-1}\right) \quad = \quad -\frac{1}{\Delta x}\left(\frac{u_i u_{i+1}}{2} - \frac{u_i u_{i-1}}{2}\right)$$

so

$$F_{i+\frac{1}{2}} = \frac{1}{2}u_i u_{i+1}$$

c) Show that the differential-difference equation (ii.a) does not conserve $< u^2 >$. You should arrive at the result

$$\sum_i \frac{1}{2}\partial_t u_i^2 = \sum_i \frac{1}{2\Delta x} u_i u_{i+1}(u_{i+1} - u_i)$$

Answer:

$$\begin{aligned}
\sum_i \frac{1}{2}\partial_t u_i^2 &= \sum_i u_i \partial_t u_i \\
&= -\sum_i u_i u_i \frac{1}{2\Delta x}\left(u_{i+1} - u_{i-1}\right) \\
&= -\sum_i u_i u_i \frac{1}{2\Delta x} u_{i+1} + \sum_i u_i u_i \frac{1}{2\Delta x} u_{i-1} \\
&= -\sum_i u_i u_i \frac{1}{2\Delta x} u_{i+1} + \sum_i u_{i+1} u_{i+1} \frac{1}{2\Delta x} u_i \\
&= \sum_i \frac{1}{2\Delta x} u_i u_{i+1}(u_{i+1} - u_i)
\end{aligned}$$

d) Time discretize the differential-difference equation using the forward method.
Answer:

$$\frac{1}{\Delta t}\left(u_i^{n+1} - u_i^n\right) = -u_i^n \frac{1}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right)$$

Use the energy method to derive the numerical stability criteria of the for this discretization. The result takes the form

$$(1 - C_i^*)^2 \leq 1$$

where $C_i^* = \frac{\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)$ is a proxy Courant number.
Answer:

$$u_i^{n+1} = u_i^n - u_i^n \frac{\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right)$$

$$
\begin{aligned}
\left(u_i^{n+1}\right)^2 &= \left(u_i^n - u_i^n \frac{\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right)\right)^2 \\
&= \left(u_i^n\right)^2 - 2\left(u_i^n\right)^2 \frac{\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) + \left(u_i^n\right)^2 \left(\frac{\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)\right)^2 \\
&= \left(1 - \frac{\Delta t}{\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) + \left(\frac{\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)\right)^2\right)\left(u_i^n\right)^2 \\
&= \left(1 - \frac{\Delta t}{\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{\Delta t^2}{4\Delta x^2}(u_{i+1}^n - u_{i-1}^n)^2\right)\left(u_i^n\right)^2 \\
&= \left(1 - \frac{\Delta t \Delta u}{\Delta x} + \frac{\Delta t^2 \Delta u^2}{4\Delta x^2}\right)\left(u_i^n\right)^2 \\
&= \left(1 - C_i^*\right)^2 \left(u_i^n\right)^2
\end{aligned}
$$

e) Write a Matlab script to solve the discrete Burger's equation (ii.d) using an initial condition of $u(x, t = 0) = \sin(2\pi x)$, $\Delta x = 1/50$ and $\Delta t = 1/1000$. Plot the solution, $u(x)$, at the two times $t = 0.15$ and $t = 0.2$. Plot the evolution of $< u^2 >$ for the interval $t = 0. \dots 0.2$

iii) Burgers equation can be written in flux form as

$$\partial_t u + \frac{1}{2}\partial_x u^2 = 0.$$

and a corresponding flux-form differential-difference equation is

$$\partial_t u = -\frac{1}{\Delta x}\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right) \quad \text{with} \quad F_{i+\frac{1}{2}} = \frac{1}{4}\left((u_i)^2 + (u_{i+1})^2\right)$$

a) Show that the differential-difference equation (iii) does not conserve $< u^2 >$. You should arrive at the result

$$\sum_i \frac{1}{2}\partial_t u_i^2 = -\sum_i \frac{1}{4} u_i u_{i+1}(u_{i+1} - u_i)$$

Answer:

$$
\begin{aligned}
\sum_i \frac{1}{2}\partial_t u_i^2 &= \sum_i u_i \partial_t u_i \\
&= -\sum_i u_i \frac{1}{4\Delta x}\left((u_{i+1})^2 - (u_{i-1})^2\right) \\
&= -\sum_i u_i \frac{1}{4\Delta x} u_{i+1} u_{i+1} + \sum_i u_i \frac{1}{4\Delta x} u_{i-1} u_{i-1}
\end{aligned}
$$

$$
\begin{aligned}
&= -\sum_i u_i \frac{1}{4\Delta x} u_{i+1} u_{i+1} + \sum_i u_{i+1} \frac{1}{4\Delta x} u_i u_i \\
&= -\sum_i \frac{1}{4\Delta x} u_i u_{i+1} (u_{i+1} - u_i)
\end{aligned}
$$

b) Using the forward method, solve the discrete model (form iii) in Matlab and plot the solution as before at $t = .15$, $t = 0.2$ and the evolution of $< u^2 >$.

c) Noting the difference in the answers to (ii.c) and (iii.a), combine the two flux forms, (ii.b) and (iii), so that the corresponding differential-difference equations conserves both $< u >$ and $< u^2 >$.

Answer: We write the flux in the form

$$
F_{i+\frac{1}{2}} = \frac{\alpha}{2} u_i u_{i+1} + \frac{(1-\alpha)}{4} \left((u_i^n)^2 + (u_{i+1}^n)^2 \right)
$$

and find α:

$$
\alpha = \frac{1}{3}
$$

ensures that the source/sinks of variance from the two schemes cancel.

d) Implement this form (iii.c) in your Matlab script and plot the solution and evolution of $< u^2 >$ as before. Why is $< u^2 >$ not constant?