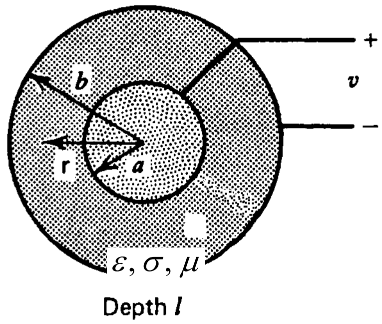
Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.013 Electromagnetics and Applications

Problem Set #4 | Issued: 9/27/05
Fall Term 2005 | Due: 10/5/05

Reading Assignment: Sections 4.1, 7.1-7.4

Problem 4.1

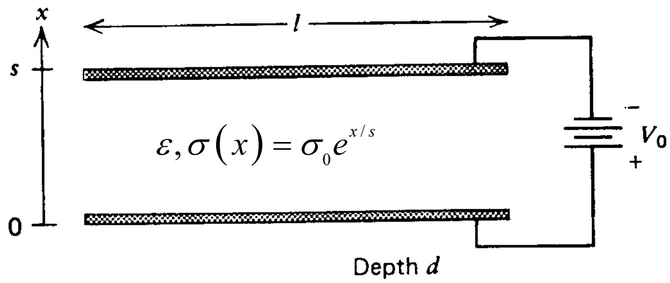


Perfectly conducting coaxial cylindrical electrodes of length l, inner radius a, and outer radius b are shown above. The material between the electrodes has dielectric permittivity ε, Ohmic conductivity σ, and magnetic permeability μ. Note that there is no electric field in the region $r<a$. Parts a-c are electroquasistatic (EQS) and parts d-f are magnetoquasistatic (MQS).

a. If there is no volume charge in the material between cylindrical electrodes, $a < r < b$, use Gauss' law in cylindrical coordinates assuming that $\overline{E} = E_r(r,t)\overline{i}_r$ to show that $E_r(r,t)$ must be of the form $E_r(r,t) = \frac{A(t)}{r}$ where $A(t)$ does not depend on r.
b. If a voltage $v(t)$ is applied across the cylindrical electrodes, what is $A(t)$?
c. What is the resistance R and capacitance C of the coaxial cylindrical structure?
d. If $\sigma = 0$ and if the frequency is low enough so that the displacement current density, $\frac{\partial \overline{D}}{\partial t}$, is negligible, then in the material between cylindrical electrodes $\nabla \times \overline{H} = 0$. For $\overline{H} = H_\phi(r,t)\overline{i}_\phi$ show that $H_\phi(r,t) = \frac{A(t)}{r}$.
e. If a total current $I(t)$ flows in the axial direction on the inner cylinder surface at r = a and returns as a surface current flowing in the opposite direction on the outer cylinder at $r = b$, find $A(t)$.
f. What is the inductance L of the coaxial cylindrical structure?
g. Calculate the product of resistance and capacitance, RC, and compare to the RC product for parallel plate electrodes.
h. Calculate the product of inductance and capacitance, LC, and compare to the LC product for parallel plate electrodes. How is the LC product related to the speed of electromagnetic waves in the material?

Problem 4.2

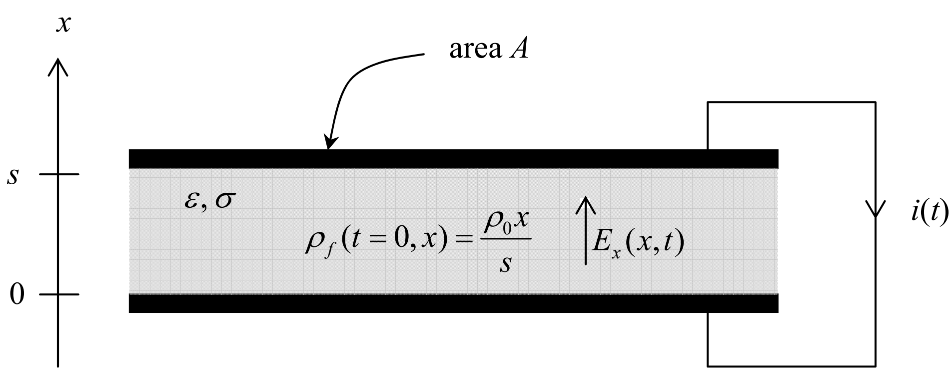
A pair of parallel plate electrodes with spacing s and voltage difference V_0 enclose an Ohmic material whose conductivity varies with position as $\sigma = \sigma_0 e^{x/s}$. The permittivity ε of the material is a constant and the system is in the DC steady state.



Adapted from Problem 3.26 in *Electromagnetic Field Theory: A Problem Solving Approach,* by Markus Zahn, 1987. Used with permission.

a. Find the electric field and the resistance.
b. What are the volume and surface charge distributions?
c. What is the total charge in the system, i.e., what is the sum of the total surface charge on the electrodes and the total volume charge in the material?

Problem 4.3



Short circuited parallel plate electrodes of area A enclose a lossy dielectric of thickness s with dielectric permittivity ε and Ohmic conductivity σ. The lossy dielectric at time $t=0$ has a free volume charge density $\rho_f(t=0,x) = \rho_0 x / s$. Neglect fringing field effects.

a. What is the volume charge distribution for $0<x<s$ as a function of time?
b. What is the electric field $E_x(x,t)$?
c. What are the surface charge densities as a function of time at $x=0$ and $x=s$?
d. What is the current $i(t)$ flowing through the short circuit?

Problem 4.4

Consider an electric scalar potential in Cartesian coordinates that only depends on coordinates x and y and that can be expressed as a product solution

$$\Phi(x, y) = X(x)Y(y) \tag{1}$$

In the region of interest, the volume charge density is zero, so the potential $\Phi(x, y)$ satisfies Laplace's equation

$$\nabla^2\Phi(x, y) = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \tag{2}$$

a. Using (1) in (2) show that (2) reduces to

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0 \tag{3}$$

Since each term in (3) is a function of x only or a function of y only, argue that each term can at most be a constant $\pm k^2$ where k^2 is called the separation constant

$$\frac{1}{X}\frac{d^2X}{dx^2} = +k^2, \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = -k^2 \tag{4}$$

b. Find solutions for X and Y when $k^2 = 0$. These are called zero separation constant solutions. Write down the general solution for $\Phi(x, y) = X(x)Y(y)$ for the zero separation constant solutions.

c. A hyperbolically shaped electrode whose surface shape obeys the equation $xy = ab$ is at potential V_0 and is placed above a grounded right-angle corner as in the figure below.

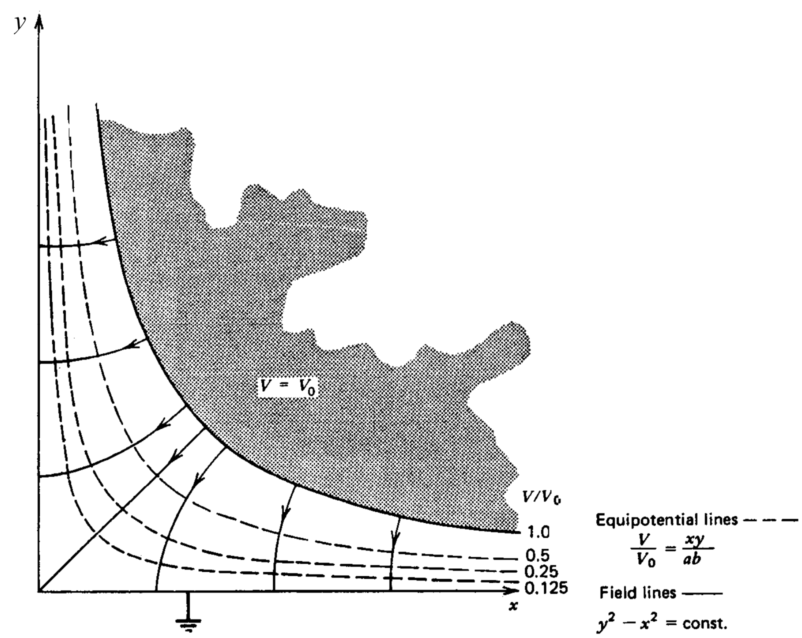


Figure 4.1 in *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

The equipotential and field lines for a hyperbolically shaped electrode $(xy = ab)$ at potential V_0 above a right-angle conducting corner are orthogonal hyperbolas.

The boundary conditions are

$$\Phi(x=0)=0,\ \Phi(y=0)=0,\ \Phi(xy=ab)=V_0 \tag{5}$$

Using the zero separation constant solutions of part (b) find the electric scalar potential $\Phi(x,y)$ that satisfies the boundary conditions.

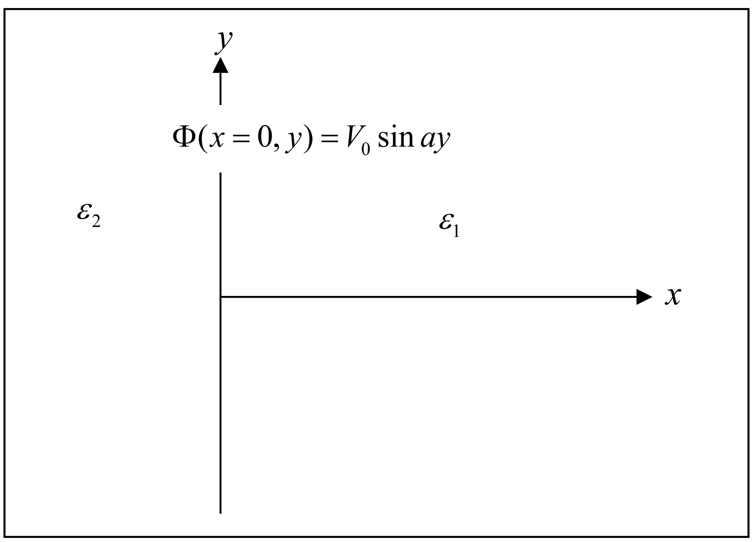
d. Find the electric field, $\bar{E}=-\nabla\Phi$ and the surface charge distribution along the *x=0* and *y=0* planes.

e. Find the equation *y(x)* of the electric field line that passes through the point (x_0, y_0).

   Hint: $\dfrac{dy}{dx}=\dfrac{E_y}{E_x}$

f. Now find the non-zero separation constant $(k^2 \neq 0)$ solutions to eq. (4) and write down the non-zero general solution for $\Phi(x,y)$ with spatially periodic solutions in the *y* direction, and exponential solutions in the *x* direction.

g.



A potential sheet with electric scalar potential

$\Phi(x=0,y)=V_0\sin ay$

is placed at *x*=0 separating dielectric media with permittivity ε_1 for *x>0* and ε_2 for *x<0.*

Find the electric scalar potential for $-\infty < x < \infty$ where $\Phi(x=\pm\infty, y)=0$.

h. For the solution of part (g), find the electric field, $\bar{E}(x,y)=-\nabla\Phi(x,y)$, the surface charge density σ_s at *x=0* on the potential sheet, and the equation of the electric field line for *x>0* that passes through the point (x_0, y_0).

   Hint: Evaluate: $\int \cot(ay)dy=\dfrac{1}{a}\ln[\sin(ay)]$