Problem 6.1

A

$$\left.\begin{aligned} \partial(\mathbf{J}_f \cdot \mathbf{J}_f)/\partial t &= 2\mathbf{J}_f \cdot \partial \mathbf{J}_f/\partial t \\ \partial \mathbf{J}_f/\partial t &= \varepsilon \omega_p^2 \mathbf{E} \end{aligned}\right\} \Longrightarrow \frac{\partial}{\partial t}(\mathbf{J}_f \cdot \mathbf{J}_f) = 2\mathbf{J}_f \cdot \varepsilon \omega_p^2 \mathbf{E}$$

Noting that $\mathbf{E} \cdot \mathbf{J}_f = \partial w_k/\partial t$, we have that

$$\frac{\partial}{\partial t}(\mathbf{J}_f \cdot \mathbf{J}_f) = 2\varepsilon\omega_p^2 \frac{\partial w_k}{\partial t} \Longleftrightarrow \frac{\partial w_k}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\mathbf{J}_f \cdot \mathbf{J}_f}{2\varepsilon \omega_p^2}\right) \Longrightarrow w_k = \frac{\mathbf{J}_f \cdot \mathbf{J}_f}{2\varepsilon\omega_p^2} = \frac{m\mathbf{J}_f \cdot \mathbf{J}_f}{2q^2 n} \text{ since } \omega_p^2 = \frac{q^2 n}{m\varepsilon}$$

B

$$\mathbf{J}_f = \rho \mathbf{v} = nq\mathbf{v} \Longleftrightarrow \mathbf{v} = \frac{\mathbf{J}_f}{nq}$$

C

$$w_k = \frac{m\mathbf{J}_f \cdot \mathbf{J}_f}{2q^2 n} = \frac{m(nq|\mathbf{v}|)^2}{2q^2 n} = \frac{1}{2} mn|\mathbf{v}|^2$$

Therefore w_k is the kinetic energy of the plasma charges.

D

$$\frac{\partial}{\partial t}(\mathbf{J}_f \cdot \mathbf{J}_f) = \varepsilon\omega_p^2 \mathbf{E} \Longrightarrow j\omega \hat{\mathbf{J}}_f = \varepsilon \omega_p^2 \hat{\mathbf{E}} \Longleftrightarrow \hat{\mathbf{J}}_f = -j\frac{\omega_p^2}{\omega}\varepsilon\hat{\mathbf{E}}. \qquad (1)$$

Maxwell's equations in complex form are:

$$\left\{\begin{aligned} \nabla \times \hat{\mathbf{E}} &= -j\omega\mu\hat{\mathbf{H}} && (2) \\ \nabla \times \hat{\mathbf{H}} &= \hat{\mathbf{J}}_f + j\omega\varepsilon\hat{\mathbf{E}} && (3) \\ \nabla \cdot \varepsilon\hat{\mathbf{E}} &= \mathbf{0} \\ \nabla \cdot \mu\hat{\mathbf{H}} &= \mathbf{0} \end{aligned}\right. \overset{(1)}{\Longleftrightarrow} \nabla \times \hat{\mathbf{H}} = -j\frac{\omega_p^2}{\omega}\varepsilon\hat{\mathbf{E}} + j\omega\varepsilon\hat{\mathbf{E}} \Longleftrightarrow \left\{\begin{aligned} \nabla \times \hat{\mathbf{E}} &= -j\omega\mu\hat{\mathbf{H}} \\ \nabla \times \hat{\mathbf{H}} &= j\omega\varepsilon\left(1 - \tfrac{\omega_p^2}{\omega^2}\right)\hat{\mathbf{E}} \\ \nabla \cdot \varepsilon\hat{\mathbf{E}} &= \mathbf{0} \\ \nabla \cdot \mu\hat{\mathbf{H}} &= \mathbf{0}. \end{aligned}\right.$$

E

Using (2) and (3) we have

$$\nabla\frac{1}{2}(\hat{\mathbf{E}}\times\hat{\mathbf{H}}^*) = \frac{1}{2}(\hat{\mathbf{H}}^*\cdot\nabla\times\hat{\mathbf{E}} - \hat{\mathbf{E}}\cdot\nabla\times\hat{\mathbf{H}}^*) = \frac{1}{2}[\hat{\mathbf{H}}^*\cdot(-j\omega\mu\hat{\mathbf{H}}) - \hat{\mathbf{E}}\cdot(\hat{\mathbf{J}}_f^* - j\omega\varepsilon\hat{\mathbf{E}}^*)]$$
$$= -2j\omega\left(\frac{1}{4}\mu|\hat{\mathbf{H}}|^2 - \frac{1}{4}\varepsilon|\hat{\mathbf{E}}|^2\right) - \frac{1}{2}\hat{\mathbf{E}}\cdot\hat{\mathbf{J}}_f^*$$
$$\Longleftrightarrow \nabla\cdot\hat{\mathbf{S}} + 2j\omega\langle w_{EM}\rangle = -\frac{1}{2}\hat{\mathbf{E}}\cdot\hat{\mathbf{J}}_f^*$$

where

$$\hat{\mathbf{S}} = \frac{1}{2}\hat{\mathbf{E}}\times\hat{\mathbf{H}}^*, \text{ and } \langle w_{EM}\rangle = \frac{1}{4}\mu|\hat{\mathbf{H}}|^2 - \frac{1}{4}\varepsilon|\hat{\mathbf{E}}|^2 \qquad (4)$$

But from (1):

$$\frac{1}{2}\hat{\mathbf{E}}\cdot\hat{\mathbf{J}}_f^* = \frac{1}{2}\hat{\mathbf{E}}\cdot\left(j\frac{\omega_p^2}{\omega}\hat{\mathbf{E}}^*\right) = 2j\omega\left(\frac{1}{4}\varepsilon\frac{\omega_p^2}{\omega^2}|\hat{\mathbf{E}}|^2\right),$$

so

$$\nabla\cdot\hat{\mathbf{S}} + 2j\omega(\langle w_{EM}\rangle + \langle w_k\rangle) = 0,$$

where

$$\langle w_k\rangle = \frac{1}{4}\varepsilon\frac{\omega_p^2}{\omega^2}|\hat{\mathbf{E}}|^2. \qquad (5)$$

F

From (4) and (5):

$$\langle w_{EM}\rangle + \langle w_k\rangle = \frac{1}{4}\mu|\hat{\mathbf{H}}|^2 - \frac{1}{4}\varepsilon|\hat{\mathbf{E}}|^2 + \frac{1}{4}\varepsilon\frac{\omega_p^2}{\omega^2}|\hat{\mathbf{E}}|^2$$
$$\Longleftrightarrow \langle w_{EM}\rangle + \langle w_k\rangle = \frac{1}{4}\mu|\hat{\mathbf{H}}|^2 - \frac{1}{4}\varepsilon(\omega)|\hat{\mathbf{E}}|^2,$$

where

$$\varepsilon(\omega) = \varepsilon\left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

which is the result we found in Problem 5.3b.

Problem 6.2

A

Since neither ε, μ nor the excitation depend on x, y, we can set $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$ and then:

$$\nabla\times\mathbf{E} = -\frac{\partial}{\partial t}\mu\mathbf{H} \Longleftrightarrow \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = -\frac{\partial}{\partial t}\mu(\hat{\mathbf{y}}H_y) \Longleftrightarrow \frac{\partial E_x}{\partial z} = -\mu\frac{\partial H_y}{\partial t} \quad (1)$$

$$\nabla \times \mathbf{H} = \frac{\partial}{\partial t}\varepsilon \mathbf{E} \Longleftrightarrow \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & H_y & 0 \end{vmatrix} = \frac{\partial}{\partial t}\varepsilon(\hat{\mathbf{x}}E_x) \Longleftrightarrow \frac{\partial H_y}{\partial z} = -\varepsilon \frac{\partial E_x}{\partial t} \qquad (2)$$

$$\left.\begin{array}{lll} \frac{\partial}{\partial t}(1) \implies & \frac{\partial^2 E_x}{\partial t \partial z} & = -\mu(z)\frac{\partial^2 H_y}{\partial t^2} \\ \frac{\partial}{\partial z}(2) \implies & \frac{\partial}{\partial z}\left(\frac{1}{\varepsilon(z)}\frac{\partial H_y}{\partial z}\right) & = -\frac{\partial^2 E_x}{\partial z \partial t} \end{array}\right\} \implies \varepsilon(z)\frac{\partial}{\partial z}\left[\frac{1}{\varepsilon(z)}\frac{\partial H_y}{\partial z}\right] - \varepsilon(z)\mu(z)\frac{\partial^2 H_y}{\partial t^2} = 0 \qquad (3)$$

B

Using now $\varepsilon(z) = \varepsilon_a e^{+\alpha z}$ and $\mu(z) = \mu_a e^{-\alpha z}$ we get:

$$\begin{aligned} (3) \implies & \frac{\partial^2 H_y}{\partial z^2} + \varepsilon(z)\frac{d}{dz}\left(\frac{1}{\varepsilon(z)}\right)\frac{\partial H_y}{\partial z} - \varepsilon(z)\mu(z)\frac{\partial H_y}{\partial t} = 0 \\ \Longleftrightarrow & \frac{\partial^2 H_y}{\partial z^2} - \frac{\varepsilon'(z)}{\varepsilon(z)}\frac{\partial H_y}{\partial z} - \varepsilon(z)\mu(z)\frac{\partial^2 H_y}{\partial t^2} = 0 \\ \Longleftrightarrow & \frac{\partial^2 H_y}{\partial z^2} - \frac{\varepsilon_a \alpha e^{+\alpha z}}{\varepsilon_a e^{+\alpha z}}\frac{\partial H_y}{\partial z} - \varepsilon_a e^{+\alpha z}\mu_a e^{-\alpha z}\frac{\partial^2 H_y}{\partial t^2} \\ \Longleftrightarrow & \frac{\partial^2 H_y}{\partial z^2} - \alpha\frac{\partial H_y}{\partial z} - \varepsilon_a\mu_a\frac{\partial^2 H_y}{\partial t^2} = 0 \qquad (4) \end{aligned}$$

Therefore $\beta = \alpha$ and $\gamma = \varepsilon_a\mu_a$.

C

Trying now the solution $\mathbf{H} = Re\left[\hat{H}_y e^{j(\omega t + \kappa z)}\hat{\mathbf{y}}\right]$ we get

$$\begin{aligned} (4) \implies & (+\kappa)^2\hat{H}_y - \alpha(+\kappa)\hat{H}_y - \varepsilon_a\mu_a(j\omega)^2\hat{H}_y = 0 \\ \Longleftrightarrow & (\kappa^2 - \alpha\kappa + \omega^2\varepsilon_a\mu_a)\hat{H}_y = 0 \\ \implies & \kappa_1 = \frac{\alpha + \sqrt{\alpha^2 - 4\omega^2\varepsilon_a\mu_a}}{2},\ \kappa_2 = \frac{\alpha - \sqrt{\alpha^2 - 4\omega^2\varepsilon_a\mu_a}}{2} \qquad (5) \end{aligned}$$

D

Boundary Conditions:

- @ $z = 0 : \hat{\mathbf{z}} \times [\hat{\mathbf{H}}(z = 0^+) - \underbrace{\hat{\mathbf{H}}(z = 0^-)}_{0}] = \hat{\mathbf{K}} \Longleftrightarrow -\hat{H}_y(z = 0^+) = K_0$ (6)
- @ $z = d : \hat{\mathbf{z}} \times [\underbrace{\hat{\mathbf{H}}(z = d^+)}_{0} - \hat{\mathbf{H}}(z = d^-)] = 0 \Longleftrightarrow \hat{H}_y(z = d^-) = 0$ (7)

E

The general solution is of the form

$$\hat{H}_y = \hat{H}_1 e^{\kappa_1 z} + \hat{H}_2 e^{\kappa_2 z},$$

so applying the boundary conditions we get

$$(6) \implies -(\hat{H}_1 + \hat{H}_2) = K_0 \quad \Longleftrightarrow \hat{H}_1 + \hat{H}_2 = -K_0$$

$$(7) \implies \hat{H}_1 e^{\kappa_1 d} + \hat{H}_2 e^{\kappa_2 d} = 0 \quad \Longleftrightarrow \hat{H}_2 = -\hat{H}_1 e^{(\kappa_1 - \kappa_2)d}$$

Together, these give

$$\hat{H}_1[1 - e^{(\kappa_1-\kappa_2)d}] = -K_0 \Longleftrightarrow \hat{H}_1 = \frac{-K_0}{1 - e^{(\kappa_1-\kappa_2)d}}$$

and

$$\hat{H}_2 = +\frac{K_0}{1 - e^{(\kappa_1-\kappa_2)d}} e^{(\kappa_1-\kappa_2)d} = +\frac{K_0}{e^{-(\kappa_1-\kappa_2)d} - 1} = \frac{-K_0}{1 - e^{-(\kappa_1-\kappa_2)d}},$$

where $\kappa_1 - \kappa_2 = \sqrt{\alpha^2 - 4\omega^2\varepsilon_a\mu_a} \equiv \Delta$ from (5).

F

Therefore

$$\hat{H}_y = -\left[\frac{e^{\kappa_1 z}}{1 - e^{(\kappa_1-\kappa_2)d}} + \frac{e^{\kappa_2 z}}{1 - e^{-(\kappa_1-\kappa_2)d}}\right] K_0,$$

and from (2)

$$\frac{\partial \hat{H}_y}{\partial z} = -j\omega\varepsilon(z)\hat{E}_x$$

$$\Longrightarrow \hat{E}_x = -\frac{-1}{j\omega\varepsilon_a e^{+\alpha z}}\left[\frac{\kappa_1 e^{\kappa_1 z}}{1 - e^{(\kappa_1-\kappa_2)d}} + \frac{\kappa_2 e^{\kappa_2 z}}{1 - e^{-(\kappa_1-\kappa_2)d}}\right] K_0 = +\frac{K_0}{j\omega\varepsilon_a}\left[\frac{\kappa_1 e^{\Delta z/2}}{1 - e^{\Delta d}} + \frac{\kappa_2 e^{-\Delta z/2}}{1 - e^{-\Delta d}}\right] e^{-\alpha z/2}$$

Problem 6.3

A

$$|\mathbf{k}|^2 = \omega^2\varepsilon_0\mu_0 \implies k_y^2 + k_z^2 = \omega^2\varepsilon_0\mu_0 \Longleftrightarrow k_z = \pm\sqrt{\omega^2\varepsilon_0\mu_0 - k_y^2} \qquad (1)$$

B

$$\nabla \times \hat{\mathbf{H}} = j\omega\varepsilon_0\hat{\mathbf{E}} \Longleftrightarrow \hat{\mathbf{y}}\frac{\partial \hat{H}_x}{\partial z} - \hat{\mathbf{z}}\frac{\partial \hat{H}_x}{\partial y} = j\omega\varepsilon_0\hat{\mathbf{E}}$$

$$\Longrightarrow \hat{\mathbf{E}} = \frac{\hat{H}_1}{j\omega\varepsilon_0}\left[\hat{\mathbf{y}}(-jk_z) - \hat{\mathbf{z}}(-jk_y)\right] e^{-j(k_y y + k_z z)} = \frac{-\hat{H}_1}{\omega\varepsilon_0}(\hat{\mathbf{y}}k_z - \hat{\mathbf{z}}k_y)e^{-j(k_y y + k_z z)},\ z > 0$$

and

$$\hat{\mathbf{E}} = \frac{\hat{H}_2}{j\omega\varepsilon_0}\left[\hat{\mathbf{y}}(jk_z) - \hat{\mathbf{z}}(-jk_y)\right] e^{-j(k_y y - k_z z)} = \frac{\hat{H}_2}{\omega\varepsilon_0}(\hat{\mathbf{y}}k_z + \hat{\mathbf{z}}k_y)e^{-j(k_y y - k_z z)},\ z < 0.$$

C

Boundary conditions at $z = 0$:

- $\hat{E}_y(z = 0^+) = \hat{E}_y(z = 0^-) \Longleftrightarrow \dfrac{\hat{H}_2}{\omega\varepsilon_0}k_z e^{-jk_y y} = -\dfrac{\hat{H}_1}{\omega\varepsilon_0}k_z e^{-jk_y y} \implies \hat{H}_2 = -\hat{H}_1 \qquad (2)$
- $\varepsilon_0\hat{E}(z = 0^+) - \varepsilon_0\hat{E}(z = 0^-) = \hat{\sigma}_0 e^{-jk_y y} \Longleftrightarrow \varepsilon_0\dfrac{\hat{H}_2}{\omega\varepsilon_0}k_y - \varepsilon_0\dfrac{-\hat{H}_1}{\omega\varepsilon_0}(-k_y) = \hat{\sigma}_0 \implies \hat{H}_2 - \hat{H}_1 = \dfrac{\omega\hat{\sigma}_0}{k_y} \qquad (3)$

From (2) and (3), we have that

$$\hat{H}_2 = -\hat{H}_1 = \frac{\omega\hat{\sigma}_0}{2k_y}.$$

D

For the fields to be evanescent in z, k_z must be imaginary, so from (1)

$$\omega^2\varepsilon_0\mu_0 - k_y^2 < 0 \implies \omega < \frac{k_y}{\sqrt{\varepsilon_0\mu_0}} = ck_y.$$

E

$$\hat{\mathbf{K}} = \hat{\mathbf{z}} \times \left[\hat{\mathbf{H}}(z=0^+) - \hat{\mathbf{H}}(z=0^-)\right] = \hat{\mathbf{y}}(\hat{H}_2 - \hat{H}_1)e^{-jk_y y} \overset{(5)}{=} \hat{\mathbf{y}}\frac{\omega\hat{\sigma}_0}{k_y}e^{-jk_y y}.$$

Problem 6.4

A

The TM reflection coefficient is

$$R = \frac{\eta_i\cos\theta_i - \eta_t\cos\theta_t}{\eta_i\cos\theta_i + \eta_t\cos\theta_t},$$

so for no reflection:

$$R = 0 \iff \eta_i\cos\theta_i = \eta_t\cos\theta_t$$

Since $\mu_1 = \mu_2 = \mu$, we have

$$\eta_i = \sqrt{\frac{\mu}{\varepsilon_1}}, \text{ and } \eta_t = \sqrt{\frac{\mu}{\varepsilon_2}}.$$

Together, these imply

$$\sqrt{\varepsilon_2}\cos\theta_i = \sqrt{\varepsilon_1}\cos\theta_t. \qquad (1)$$

From Snell's law:

$$n_i\sin\theta_i = n_t\sin\theta_t.$$

Since $\mu_1 = \mu_2 = \mu$, $n_i = \sqrt{\varepsilon_1\mu}$ and $n_t = \sqrt{\varepsilon_2\mu}$,

$$\sqrt{\varepsilon_1}\sin\theta_i = \sqrt{\varepsilon_2}\sin\theta_t. \qquad (2)$$

From (1) $\cdot$ (2),

$$\sqrt{\varepsilon_1\varepsilon_2}\sin\theta_i\cos\theta_i = \sqrt{\varepsilon_1\varepsilon_2}\sin\theta_t\cos\theta_t \implies \sin 2\theta_i = \sin 2\theta_t$$

$$\implies \begin{cases} \theta_i &= \theta_t \text{ (cannot satisfy (1), (2))} \\ 2\theta_i &= \pi - 2\theta_t \end{cases} \implies \boxed{\theta_i + \theta_t = \frac{\pi}{2} \qquad (3)}$$

B

From (2),

$$\sqrt{\varepsilon_1}\sin\theta_i = \sqrt{\varepsilon_2}\sin\left(\frac{\pi}{2} - \theta_i\right) = \sqrt{\varepsilon_2}\cos\theta_i \implies \tan\theta_i = \boxed{\tan\theta_B = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}.} \qquad (4)$$

Note: Physical explanation for zero reflection: Physically, the transmitted wave excites dipoles inside the transmission medium into oscillation. These dipoles cannot radiate along their axis of oscillation, which lies, due to (3), in the direction of the reflected beam. Therefore the reflected beam cannot be excited and thus total transmission occurs.

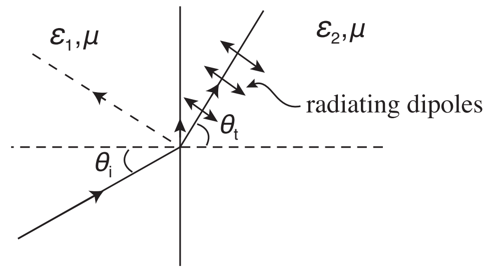

Figure 1: Dipole radiation. (Image by MIT OpenCourseWare.)

C

From (2) we see that for total internal reflection we must have:

$$\sin\theta_t \geq 1 \implies \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_i \geq 1 \implies \boxed{\sin\theta_c = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}.} \qquad (5)$$

For the critical angle to exist:

$$\sin\theta_c \leq 1 \implies \sqrt{\varepsilon_2} \leq \sqrt{\varepsilon_1} \iff n_2 \leq n_1.$$

D

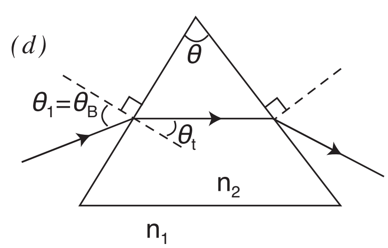

Figure 2: Refraction. (Image by MIT OpenCourseWare.)

From the small triangle:

$$\theta + 2\left(\frac{\pi}{2} - \theta_t\right) = \pi.$$

But from (3):

$$\theta_B = \frac{\pi}{2} - \theta_t.$$

Hence,

$$\theta + 2\theta_B = \pi \implies \theta = \pi - 2\theta_B. \qquad (6)$$

For $n_1 = 1$ and $n_2 = 1.45$, we get $\tan\theta_B = 1.45 \implies \theta_B \approx 0.31\pi = 55.8^\circ \overset{(6)}{\implies} \theta \approx \pi - 2\cdot 0.31\pi = 0.38\pi = 68.4^\circ$ and then $\theta_t = \pi/2 - \theta_B = 0.19\pi = 34.2^\circ$.

E

Clearly for TM light, the transmitted power equals the incident one. For TE light, the reflection coefficient is (using $\mu_1 = \mu_2 = \mu$)

$$R = \frac{\eta_t \cos\theta_i - \eta_i \cos\theta_t}{\eta_t \cos\theta_i + \eta_i \cos\theta_t} = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t} = \frac{\cos 55.8^\circ - 1.45 \cos 34.2^\circ}{\cos 55.8^\circ + 1.45 \cos 34.2^\circ} = -0.3618$$

at the input surface and $-R = 0.3618$ at the output. Therefore the reflected power at each surface is

$$|R|^2 = 0.131$$

and thus the transmitted power is

$$1 - |R|^2 = 0.869.$$

In total, the transmitted power from both surfaces

$$S_t/S_i = (1 - |R|^2)^2 = 0.755 = 75.5\%.$$