6.013/ESD.013J — Electromagnetics and Applications	Fall 2005
Problem Set 9 - Solutions	
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Problem 9.1

A

$$\mathbf{H} = -\frac{I}{2\pi r}\,\hat{\mathbf{i}}_\phi$$

$$\mathbf{B} = \begin{cases} -\frac{\mu_0 I}{2\pi r}\,\hat{\mathbf{i}}_\phi & \text{in free space} \\ -\frac{\mu I}{2\pi r}\,\hat{\mathbf{i}}_\phi & \text{in fluid} \end{cases}$$

B

$$L(h) = \frac{\ln\frac{b}{a}}{2\pi}[\mu_0(l-h) + \mu(h+s)]$$

C

$$f_x = \frac{1}{2}I^2\frac{dL(h)}{dh} = \frac{\ln\left(\frac{b}{a}\right)I^2}{4\pi}[\mu - \mu_0]$$

D

$$f_x = f_g \implies f_x = \rho_m g h\pi(b^2 - a^2),\ h = \frac{I^2\ln\left(\frac{b}{a}\right)(\mu-\mu_0)}{4\pi^2\rho_m g(b^2-a^2)}$$

Problem 9.2

A

$$\lambda_1 = L_{11}\,i_1 + L_{12}\,i_2$$
$$\lambda_2 = L_{21}\,i_1 + L_{22}\,i_2 \qquad (L_{21} = L_{12})$$

The CO-ENERGY:

$$\begin{aligned} W_m'(i_1, i_2, \theta) &= \frac{1}{2}L_{11}i_1^2 + L_{21}i_1i_2 + \frac{1}{2}L_{22}i_2^2 \\ &= \frac{1}{2}[L_0 + M\cos 2\theta]i_1^2 + M\sin 2\theta i_1 i_2 + \frac{1}{2}[L_0 - M\cos 2\theta]i_2^2 \end{aligned}$$

so that the torque of electrical origin is

$$T^e = M[\sin 2\theta(i_2^2 - i_1^2) + 2\cos 2\theta i_1 i_2]$$

B

$$i_2^2 - i_1^2 = I^2[\sin^2 \omega_s t - \cos^2 \omega_s t] = -I^2 \cos 2\omega_s t$$

$$i_1 i_2 = I^2 \sin \omega_s t \cos \omega_s t = \frac{1}{2} I^2 \sin 2\omega_s t$$

Therefore,

$$\begin{aligned} T^e &= MI^2[-\sin 2\theta \cos 2\omega_s t + \sin 2\omega_s t \cos 2\theta] \\ &= MI^2 \sin(2\omega_s t - 2\theta). \end{aligned}$$

Substitution of $\theta = \omega_m t + \gamma$ obtains

$$T^e = -MI^2 \sin[2(\omega_m - \omega_s)t + 2\gamma].$$

For constant torque $\omega_m = \omega_s$:

$$T^e_{\text{constant}} = -MI^2 \sin 2\gamma$$

C

To determine the equilibrium angle γ_0 we write the equation of motion in the steady state

$$J\ddot{\theta} = T_0 - MI^2 \sin 2\gamma_0 = 0$$

$$T_0 = MI^2 \sin 2\gamma_0 \xrightarrow[\text{Equilibrium Condition}]{} \gamma_0 = \frac{1}{2} \sin^{-1}\left(\frac{T_0}{MI^2}\right)$$

Equilibrium possible as long as $MI^2 > T_0$. Now, for the perturbation equation

$$J\ddot{\theta} = T_0 + T' - MI^2 \sin(2\gamma)$$

$$J\frac{d^2}{dt^2}(\omega_m t + \gamma_0 + \gamma') = T_0 + T' - MI^2 \sin(2\gamma_0 + 2\gamma')$$

$$J\frac{d^2}{dt^2}\gamma' = T' - (2MI^2 \cos 2\gamma_0)\gamma'.$$

The constant terms cancel by the equilibrium condition.

D

With $T'(t) = I_0 u_0(t)$,

$$\begin{aligned} \frac{d\gamma'}{dt}(0^+) &= \frac{I_0}{J} \\ \gamma'(0^+) &= 0 \end{aligned}$$

$$\gamma'(t) = \frac{I_0}{J\sqrt{\frac{2MI^2 \cos 2\gamma_0}{J}}} \sin\left[\sqrt{\frac{2MI^2 \cos 2\gamma_0}{J}}\, t\right]$$

E

The equilibrium condition is given by

$$T_0 = MI^2 \sin 2\gamma_0 \implies \gamma_0 = \frac{1}{2}\sin^{-1}\left(\frac{T_0}{MI^2}\right)$$

Graphically, we see there are two possible values of γ_0 at most. From part D, perturbation motions are stable if $\cos 2\gamma_0 > 0$, as $\gamma'(t)$ oscillates with a finite amplitude, and unstable if $\cos 2\gamma_0 < 0$, as then the oscillation frequency is imaginary and $\gamma'(t)$ grows to infinite amplitude.

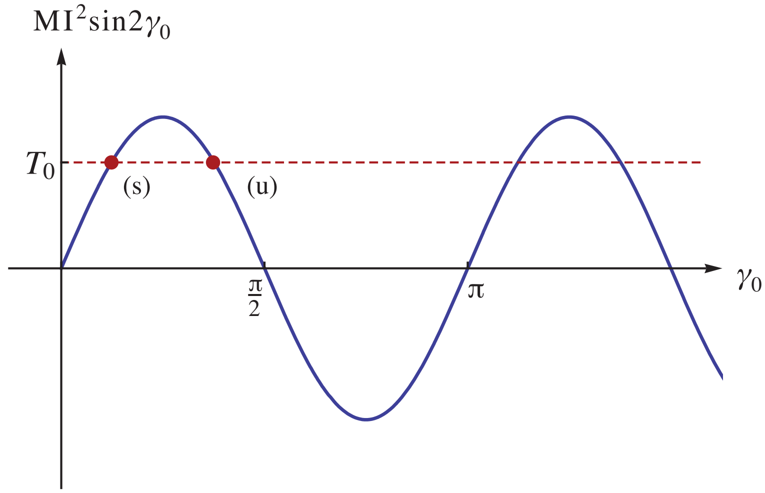

Figure 1: Plot of $MI^2 \sin 2\gamma_0$ versus γ_0 showing stable (s) and unstable (u) operating points when a mechanical torque T_0 is applied. (Image by MIT OpenCourseWare.)

The equilibrium labeled (s) is stable as the resulting torque restores the system to equilibrium with $\cos 2\gamma_0 > 0$. The point labeled (u) is unstable, as $\cos 2\gamma_0$ is negative.

Problem 9.3

A

When $\theta = 0$, there is no overlap $\implies A = 0$. When $\theta = \pi/2$, there is complete overlap $\implies A = \pi R^2/2$ and changes linearly: $A(\theta) = \theta R^2$. There are $2N-1$ pairs.

$$C = (2N-1)\theta R^2 \frac{\varepsilon_0}{g}$$
$$Q = C(\theta)V^2$$

B

$$W_e' = \frac{1}{2}C(\theta)V^2$$
$$T^e = \left.\frac{\partial W_e'}{\partial \theta}\right|_V = \frac{(2N-1)R^2\varepsilon_0 V^2}{2g}$$

C

$$J\frac{d^2\theta}{dt^2} = -K\theta - B\frac{d\theta}{dt} + \frac{1}{2}\frac{V^2(2N-1)R^2\varepsilon_0}{g}$$
$$\theta = \frac{V_0^2(2N-1)R^2\varepsilon_0}{2gK}$$

Problem 9.4

A

$$\text{Ampere's Law} \rightarrow Ni = dH_d + xH_x$$
$$\text{Gauss' Law} \rightarrow \mu_0 wbH_d = \mu_0 waH_x$$

$$\implies H_x = \frac{Ni}{\left(x + \frac{da}{b}\right)}$$

$$\lambda = N\mu_0 awH_x \implies L = \frac{N^2\mu_0 aw}{\left(x + \frac{da}{b}\right)}$$

B

$$W_m = \frac{1}{2}\frac{\lambda^2}{L} = \frac{1}{2}\frac{\lambda^2\left(\frac{da}{b} + x\right)}{N^2\mu_0 aw}$$

C

$$f^e = -\left.\frac{\partial W_m}{\partial x}\right|_\lambda = -\frac{1}{2}\frac{\lambda^2}{N^2\mu_0 aw}$$

D

$$I(t) = \frac{V}{R} + \frac{\lambda}{L} = \frac{1}{R}\frac{d\lambda}{dt} + \frac{\lambda\left(\frac{da}{b} + x\right)}{N^2\mu_0 aw}$$

E

$$M\frac{d^2x}{dt^2} = Mg - \frac{1}{2}\frac{\lambda^2}{N^2\mu_0 aw}$$