MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Markov Chains

- Problem 19 from Poisson process exercises in [BT]
- **Fact:** If there is a single recurrent class, the frequency with which the edge $i \to j$ is traversed is $\pi_i p_{ij}$. This fact allows us to solve some problems easily.
- Consider a birth-death Markov chain. State space $1, \ldots, m$. If you are at state i, you go to $i+1$ with probability d_i and $i-1$ with probability b_i. The numbers b_i, d_i are given. You stay at i with probability $1 - b_i - d_i$.

  See problem 21 in Markov chain chapter of [BT] for a picture.

  The edge $i \to i+1$ is traversed in the same proportion as the edge $i+1 \to i$, so

  $$\pi_i b_i = \pi_{i+1} d_{i+1},$$

  or

  $$\pi_{i+1} = \pi_i \frac{b_i}{d_{i+1}}.$$

  The above recursion allows one to write all the stationary probabilities in terms of π_1 as

  $$\pi_2 = \pi_1 \frac{b_1}{d_2},$$

  $$\pi_3 = \pi_2 \frac{b_2}{d_3} = \pi_1 \frac{b_1 b_2}{d_2 d_3},$$

  and finally

  $$\pi_m = \pi_{m-1} \frac{b_{m-1}}{d_m} = \pi_1 \frac{b_1 b_2 \cdots b_{m-1}}{d_2 d_3 \cdots d_m}.$$

  Together with the equation

  $$\sum_i \pi_i = 1,$$

this completely determines π_i. For example, suppose $b_i = 1/3, d_i = 2/3$ for all i. Then,

$$\pi_{i+1} = \frac{1}{2}\pi_i,$$

and so

$$\pi_1(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{m-1}}) = 1,$$

and lets assume m is very large, so that

$$\pi_1 \approx \frac{1}{2},$$

and

$$\pi_2 \approx \frac{1}{4},$$

$$\pi_3 \approx \frac{1}{8},$$

and so on.

- **Random walk on a graph.** A particle performs a random walk on the vertex set of a connected undirected graph G, which for simplicity we assume to have neither loops nor multiple edges. At each stage it moves to a neighbor of its current position, each such neighbor being chosen with equal probability. If G has $\eta < \infty$ edges, show that the stationary distribution is given by $\pi_v = d_v/(2\eta)$, where d_v is the degree of each vertex v.

  One way to do this problem is to simply check that the proposed solution satisfies the defining equations: $\pi P = \pi$, and $\sum_v \pi_v = 1$ (we can see immediately that we have nonnegativity). We have:

$$\begin{aligned} \sum_v \pi_v &= \sum_v \frac{d_v}{2\eta} \\ &= \frac{1}{2\eta}\sum_v d_v \\ &= 1, \end{aligned}$$

  since the sum of the degrees is twice the number of edges (each edge increases the sum of the degrees by exactly 2). Similarly, we can show that $\pi P = \pi$. Let us define δ_{vu} to be 1 if vertices u and v are adjacent, and 0 otherwise. Then, we have:

$$\begin{aligned}\sum_v \pi_v P_{vu} &= \frac{1}{2\eta}\sum_v d_v\left(\frac{1}{d_v}\delta_{vu}\right)\\ &= \frac{1}{2\eta}\sum_v \delta_{vu}.\end{aligned}$$

But $\sum_v \delta_{vu}$ is the number of edges incident to node u, that is, $\sum_v \delta_{vu} = d_u$. Therefore we have:

$$\sum_v \pi_v P_{vu} = \frac{1}{2\eta}d_u = \frac{d_u}{2\eta} = \pi_u.$$

This is what we wanted to show.

[**Note:** HW11-07; from [GS]]

Exercise 133. A particle performs a random walk on a bow tie $ABCDE$ drawn beneath, where C is the knot. From any vertex, its next step is equally likely to be to any neighbouring vertex. Initially it is at A. Find the expected value of:

(a) The time of first return to A.

(b) The number of visits to D before returning to A.

(c) The number of visits to C before returning to A.

(d) The time of first return to A, given that there were no visits to E before the return to A.

(e) The number of visits to D before returning to A, given that there were no visits to E before the return to A.

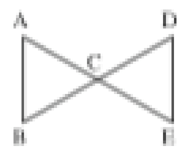

Figure 1: A simple example of the set operation we describe.

Solution: First, we can compute that the steady state distribution is $\pi_A = \pi_B = \pi_D = \pi_E = 1/6$, and $\pi_C = 1/3$. We can do this either by solving a system of linear equations (as usual) or just use our result from the first problem above.

(a) By the result from class, and on the handout, we have: $t_A = 1/\pi_A = 6$. Alternatively, we can solve the following system of equations (observe than t_A appears in only one equation):

$$\begin{aligned}
t_A &= \frac{1}{2}(t_B + 1) + \frac{1}{2}(t_C + 1) \\
t_B &= \frac{1}{2} + \frac{1}{2}(t_C + 1) \\
t_C &= \frac{1}{4} + \frac{1}{4}(t_B + 1) + \frac{1}{4}(t_D + 1) + \frac{1}{4}(t_E + 1) \\
t_D &= \frac{1}{2}(t_C + 1) + \frac{1}{2}(t_E + 1) \\
t_E &= \frac{1}{2}(t_C + 1) + \frac{1}{2}(t_D + 1).
\end{aligned}$$

(b) By the result from the handout on Markov Chains, we know that

$$\pi_D = \frac{\mathbb{E}[\#\text{ transitions to } D \text{ in a cycle that starts and ends at } A]}{\mathbb{E}[\#\text{ transitions in a cycle that starts and ends at } A]},$$

from which we find that the quantity we wish to compute is $6\pi_D = 1$.

(c) Using the same method as in part (b), we find the answer to be $6\pi_C = 2$.

(d) We let $\mathbb{P}_i(\cdot) = \mathbb{P}(\cdot|X_0 = i)$, and let T_j be the time of the first passage to state j, and let $\nu_i = \mathbb{P}_i(T_A < T_E)$. Then, as we obtained the equations above, that is, by conditioning on the first step, we have

$$\begin{aligned}
\nu_A &= \frac{1}{2}\nu_B + \frac{1}{2}\nu_C \\
\nu_B &= \frac{1}{2} + \frac{1}{2}\nu_C \\
\nu_C &= \frac{1}{4} + \frac{1}{4}\nu_B + \frac{1}{4}\nu_D \\
\nu_D &= \frac{1}{2}\nu_C.
\end{aligned}$$

Solving these, we find: $\nu_A = 5/8, \nu_B = 3/4, \nu_C = 1/2, \nu_D = 1/4$. Now we can compute the conditional transition probabilities, which we call τ_{ij}. We have:

$$\begin{aligned}
\tau_{AB} &= \mathbb{P}_A(X_1 = B|T_A < T_E) \\
&= \frac{\mathbb{P}_A(X_1 = B)P_B(T_A < T_E)}{\mathbb{P}_A(T_A < T_E)} \\
&= \frac{\nu_B}{2\nu_A} = \frac{3}{5}.
\end{aligned}$$

Similarly, we find: $\tau_{AC} = 2/5, \tau_{BA} = 2/3, \tau_{BC} = 1/3, \tau_{CA} = 1/2, \tau_{CB} = 3/8, \tau_{CD} = 1/8, \tau_{DC} = 1$. Now we have essentially reduced to a problem like part (a). We can compute the conditional expectation by solving a system of linear equations using the new transition probabilities:

$$\begin{aligned}
\tilde{t}_A &= 1 + \frac{3}{5}\tilde{t}_B + \frac{2}{5}\tilde{t}_C \\
\tilde{t}_B &= 1 + \frac{2}{3}(1) + \frac{1}{3}\tilde{t}_C \\
\tilde{t}_C &= 1 + \frac{1}{2}(1) + \frac{3}{8}\tilde{t}_B + \frac{1}{8}\tilde{t}_D \\
\tilde{t}_D &= 1 + \tilde{t}_C.
\end{aligned}$$

Solving these equations, yields $\tilde{t}_A = 14/5$.

(e) We can use the conditional transition probabilities above, to reduce to a problem essentially like that in part (b). Let N be the number of visits to D. Then, denoting by η_i the expected value of N given that we start at i, and that $T_A < T_E$, we have the equations:

$$\begin{aligned}
\eta_A &= \frac{3}{5}\eta_B + \frac{2}{5}\eta_B \\
\eta_B &= 0 + \frac{1}{3}\eta_C \\
\eta_C &= 0 + \frac{3}{8}\eta_B + \frac{1}{8}(1 + \eta_D) \\
\eta_D &= \eta_C.
\end{aligned}$$

Solving, we obtain: $\eta_A = 1/10$.

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