6.641, Electromagnetic Fields, Forces, and Motion
Prof. Markus Zahn
Lecture 2: Differential Form of Maxwell's Equations

I. Divergence Theorem

1. Divergence Operation

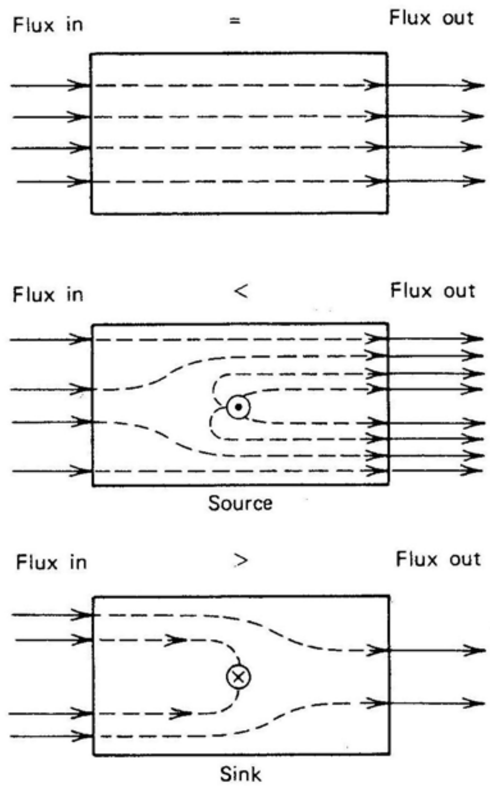


Figure 1-13 The net flux through a closed surface tells us whether there is a source or sink within an enclosed volume.

$$\oint_S \overline{A} \cdot \overline{dS} = \int_V \operatorname{div}\left(\overline{A}\right) dV$$

$$\operatorname{div} \overline{A} = \lim_{\Delta V \to 0} \frac{\oint_S \overline{A} \cdot \overline{dS}}{\Delta V}$$

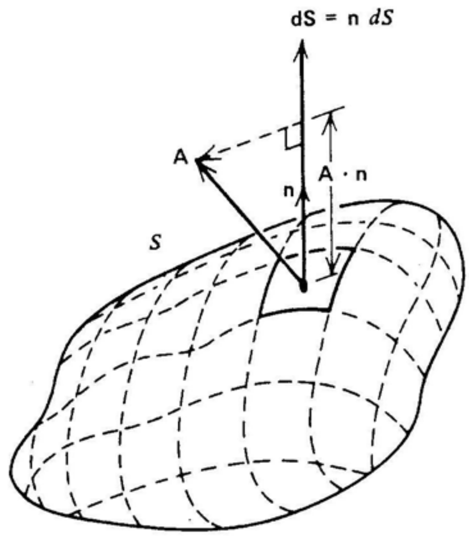


Figure 1-14 The flux of a vector **A** through the closed surface S is given by the surface integral of the component of **A** perpendicular to the surface S. The differential vector surface area element **dS** is in the direction of the unit normal **n**.

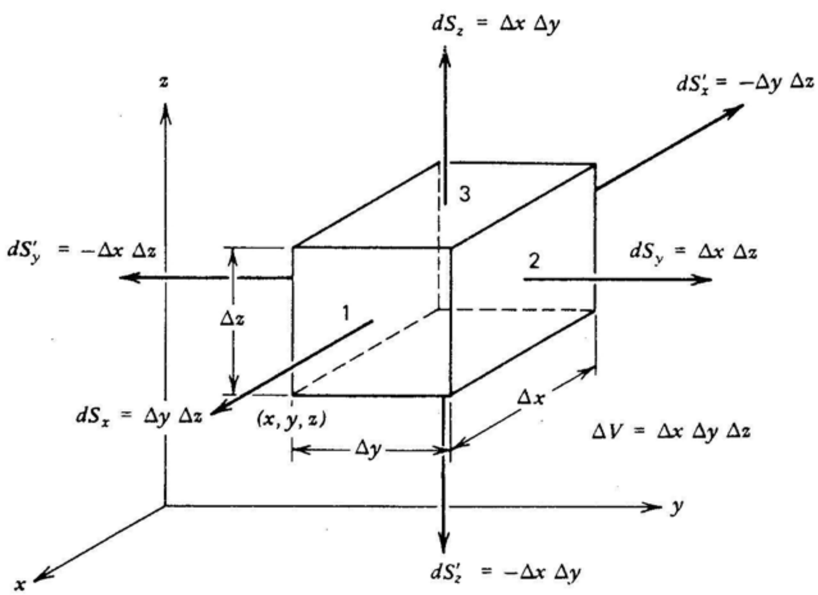


Figure 1-15 Infinitesimal rectangular volume used to define the divergence of a vector.

$$\Phi = \int_1 A_x(x,y,z)\,dydz - \int_{1'} A_x(x-\Delta x,y,z)\,dydz$$
$$+ \int_2 A_y(x,y+\Delta y,z)\,dxdz - \int_{2'} A_y(x,y,z)\,dxdz$$
$$+ \int_3 A_z(x,y,z+\Delta z)\,dxdy - \int_{3'} A_z(x,y,z)\,dxdy$$

$$\Phi \approx \Delta x \Delta y \Delta z \left\{ \frac{\left[A_x(x,y,z) - A_x(x-\Delta x,y,z)\right]}{\Delta x} + \frac{\left[A_y(x,y+\Delta y,z) - A_y(x,y,z)\right]}{\Delta y} + \frac{\left[A_z(x,y,z+\Delta z) - A_z(x,y,z)\right]}{\Delta z} \right\}$$

$$\approx \Delta V \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right]$$

$$\operatorname{div} \overline{A} = \lim_{\Delta V \to 0} \frac{\oint_S \overline{A} \cdot \overline{dS}}{\Delta V} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Del Operator: $\nabla = \bar{i}_x \dfrac{\partial}{\partial x} + \bar{i}_y \dfrac{\partial}{\partial y} + \bar{i}_z \dfrac{\partial}{\partial z}$

$$= \quad \operatorname{div} \overline{A} \quad \nabla \cdot \overline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

2. Gauss' Integral Theorem

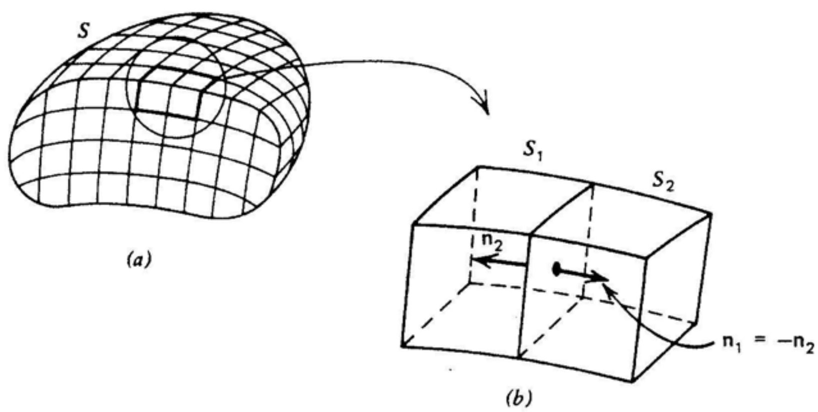


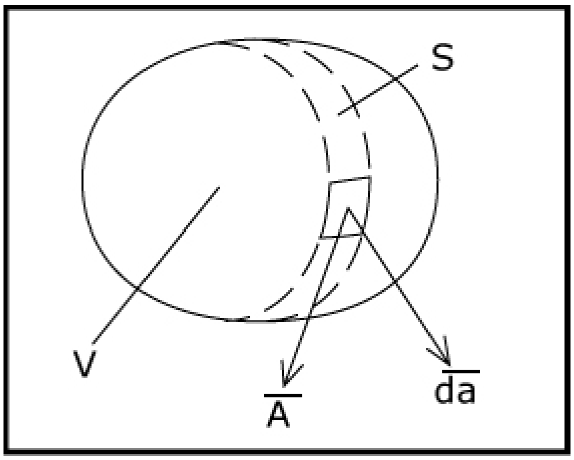
Figure 1-17 Nonzero contributions to the flux of a vector are only obtained across those surfaces that bound the outside of a volume. (a) Within the volume the flux leaving one incremental volume just enters the adjacent volume where (*b*) the outgoing normals to the common surface separating the volumes are in opposite directions.

$$\oint_S \overline{A} \bullet \overline{dS} = \sum_{\substack{i=1 \\ N\to\infty}}^{N} \oint_{dS_i} \overline{A} \bullet \overline{dS_i}$$

$$= \lim_{\substack{N\to\infty \\ \Delta V_n \to 0}} \sum_{i=1}^{N} \left(\nabla \bullet \overline{A}\right) \Delta V_i$$

$$= \int_V \nabla \bullet \overline{A}\, dV$$

$$\int_V \nabla \bullet \overline{A}\, dV = \oint_S \overline{A} \bullet \overline{da}$$



3. Gauss' Law in Differential Form

$$\oint_S \varepsilon_0\, \overline{E} \bullet \overline{da} = \int_V \nabla \bullet \left(\varepsilon_0 \overline{E}\right) dV = \int_V \rho\, dV$$

$$\nabla \bullet \left(\varepsilon_0 \overline{E}\right) = \rho$$

$$\oint_S \mu_0\, \overline{H} \bullet \overline{da} = \int_V \nabla \bullet \left(\mu_0 \overline{H}\right) dV = 0$$

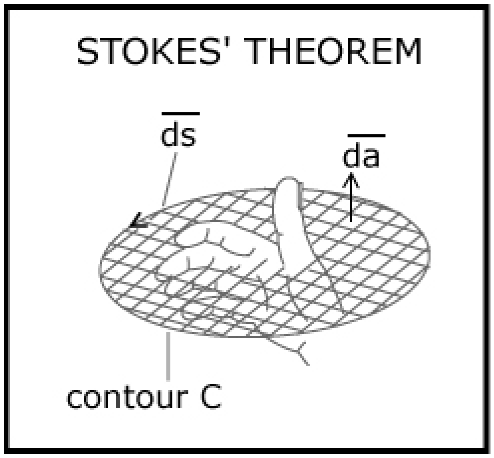
$$\nabla \bullet \left(\mu_0 \overline{H}\right) = 0$$

II. Stokes' Theorem

1. Curl Operation

$$\oint_C \overline{A} \bullet \overline{ds} = \int_S \text{Curl}\left(\overline{A}\right) \bullet \overline{da}$$

$$\text{Curl}\left(\overline{A}\right)_n = \lim_{da_n \to 0} \frac{\oint_C \overline{A} \bullet \overline{ds}}{da_n}$$



$$\int_S \nabla \times \overline{A} \cdot \overline{da} = \oint_C \overline{A} \cdot \overline{ds}$$

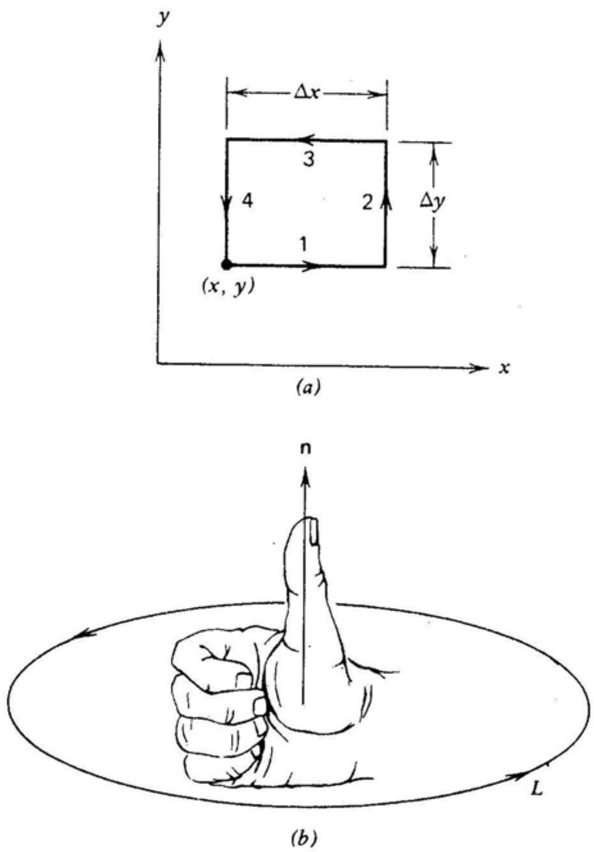


Figure 1-19 (*a*) Infinitesimal rectangular contour used to define the circulation. (*b*) The right-hand rule determines the positive direction perpendicular to a contour.

$$\oint_C \overline{A} \cdot \overline{ds} = \underset{1}{\int_x^{x+\Delta x}} A_x(x, y)\, dx + \underset{2}{\int_y^{y+\Delta y}} A_y(x + \Delta x, y)\, dy + \underset{3}{\int_{x+\Delta x}^{x}} A_x(x, y + \Delta y)\, dx$$

$$+ \underset{4}{\int_{y+\Delta y}^{y}} A_y(x, y)\, dy$$

$$= \Delta x \Delta y \left[\frac{\left[A_x(x, y) - A_x(x, y + \Delta y) \right]}{\Delta y} + \frac{\left[A_y(x + \Delta x, y) - A_y(x, y) \right]}{\Delta x} \right]$$

$$= da_z \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right]$$

$$\text{Curl}\left(\overline{A}\right)_z = \frac{\oint \overline{A} \cdot \overline{ds}}{da_z} = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

By symmetry

$$\text{Curl}\left(\overline{A}\right)_y = \frac{\oint \overline{A} \cdot \overline{ds}}{da_y} = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$$

$$\text{Curl}\left(\overline{A}\right)_x = \frac{\oint \overline{A} \cdot \overline{ds}}{da_x} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

$$\text{Curl}\, \overline{A} = \bar{i}_x \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right] + \bar{i}_y \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right] + \bar{i}_z \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right]$$

$$= \det \begin{bmatrix} \bar{i}_x & \bar{i}_y & \bar{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{bmatrix}$$

$$= \nabla \times \overline{A}$$

2. Stokes' Integral Theorem

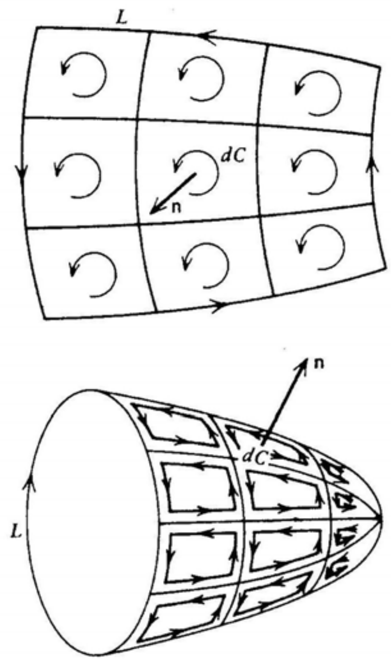


Figure 1-23 Many incremental line contours distributed over any surface, have nonzero contribution to the circulation only along those parts of the surface on the boundary contour L.

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$$\lim_{N\to\infty} \sum_{i=1}^{N} \oint_{dC_i} \overline{A} \cdot \overline{ds_i} = \oint_{C} \overline{A} \cdot \overline{ds}$$

$$= \sum_{i=1}^{N\to\infty} \left(\nabla \times \overline{A}\right) \cdot \overline{da_i}$$

$$= \int_{S} \left(\nabla \times \overline{A}\right) \cdot \overline{da}$$

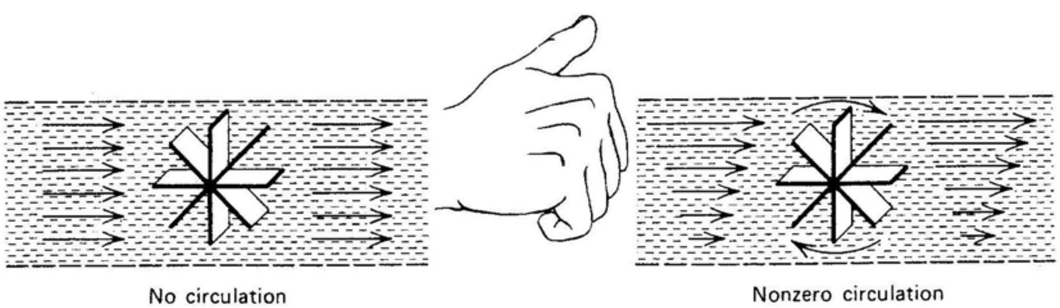

Figure 1-20 A fluid with a velocity field that has a curl tends to turn the paddle wheel. The curl component found is in the same direction as the thumb when the fingers of the right hand are curled in the direction of rotation.

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3. Faraday's Law in Differential Form

$$\oint_C \overline{E} \cdot \overline{ds} = \int_S \left(\nabla \times \overline{E}\right) \cdot \overline{da} = -\frac{d}{dt} \int_S \mu_0 \, \overline{H} \cdot \overline{da}$$

$$\nabla \times \overline{E} = -\mu_0 \frac{\partial \overline{H}}{\partial t}$$

4. Ampère's Law in Differential Form

$$\oint_C \overline{H} \cdot \overline{ds} = \int_S \nabla \times \overline{H} \cdot \overline{da} = \int_S \overline{J} \cdot \overline{da} + \frac{d}{dt} \int_S \varepsilon_0 \, \overline{E} \cdot \overline{da}$$

$$\nabla \times \overline{H} = \overline{J} + \varepsilon_0 \frac{\partial \overline{E}}{\partial t}$$

III. <u>Applications to Maxwell's Equations</u>

1. Vector Identity

$$\lim_{C \to 0} \oint_C \overline{A} \cdot \overline{ds} = 0 = \oint_S \left(\nabla \times \overline{A}\right) \cdot \overline{da} = \int_V \nabla \cdot \left(\nabla \times \overline{A}\right) dV$$

$$\nabla \cdot \left(\nabla \times \overline{A}\right) = 0$$

2. Charge Conservation

$$\nabla \cdot \left\{\nabla \times \overline{H} = \overline{J} + \varepsilon_0 \frac{\partial \overline{E}}{\partial t}\right\}$$

$$0 = \nabla \cdot \left[\overline{J} + \varepsilon_0 \frac{\partial \overline{E}}{\partial t}\right]$$

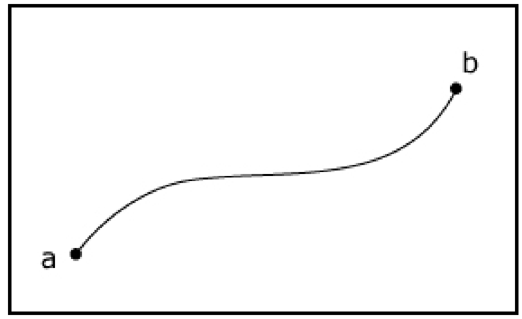
$$0 = \nabla \cdot \overline{J} + \frac{\partial \rho}{\partial t}$$

3. Magnetic Field

$$\nabla \cdot \left\{\nabla \times \overline{E} = -\mu_0 \frac{\partial \overline{H}}{\partial t}\right\}$$

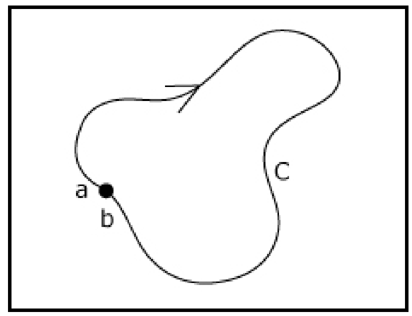
$$0 = -\frac{\partial}{\partial t}\left[\nabla \cdot \mu_0 \overline{H}\right] \Rightarrow \nabla \cdot \left(\mu_0 \overline{H}\right) = 0$$

4. Vector Identity



$$\int_a^b \bar{E} \cdot \overline{dl} = \Phi(a) - \Phi(b)$$

if a=b



$$\oint_C \bar{E} \cdot \overline{dl} = \Phi(a) - \Phi(a) = 0$$

$$\bar{E} = -\nabla\Phi$$

$$\oint_C \nabla\Phi \cdot \overline{dl} = 0$$

$$\int_S \nabla \times (\nabla f) \cdot \overline{da} = \oint_C \nabla f \cdot \overline{dl} = 0 \Rightarrow \nabla \times (\nabla f) = 0$$

IV. Summary of Maxwell's Equations in Free Space

Integral Form	Differential Form
Faraday's Law	
$\oint_C \overline{E}\cdot\overline{dl} = -\mu_0 \frac{d}{dt}\int_S \overline{H}\cdot\overline{da}$	$\nabla\times\overline{E} = -\mu_0 \frac{\partial\overline{H}}{\partial t}$
Ampere's Law	
$\oint_C \overline{H}\cdot\overline{dl} = \int_S \overline{J}\cdot\overline{da} + \varepsilon_0 \frac{d}{dt}\int_S \overline{E}\cdot\overline{da}$	$\nabla\times\overline{H} = \overline{J} + \varepsilon_0 \frac{\partial\overline{E}}{\partial t}$
Gauss' Law	
$\oint_S \varepsilon_0\overline{E}\cdot\overline{da} = \int_V \rho\, dV$	$\nabla\cdot\left(\varepsilon_0\overline{E}\right) = \rho$
$\oint_S \mu_0\overline{H}\cdot\overline{da} = 0$	$\nabla\cdot\left(\mu_0\overline{H}\right) = 0$
Conservation of charge	
1. $\oint_C \overline{J}\cdot\overline{da} + \frac{d}{dt}\int_V \rho\, dV = 0$	$\nabla\cdot\overline{J} + \frac{\partial\rho}{\partial t} = 0$
2. $\oint_S \left[\overline{J} + \varepsilon_0\frac{\partial\overline{E}}{\partial t}\right]\cdot\overline{da} = 0$	$\nabla\cdot\left[\overline{J} + \varepsilon_0\frac{\partial\overline{E}}{\partial t}\right] = 0$

EQS Limit	MQS Limit
$\nabla\times\overline{E}\approx 0,\ \overline{E} = -\nabla\Phi$	$\nabla\times\overline{E} = -\mu_0\frac{\partial\overline{H}}{\partial t}$
$\nabla\cdot\overline{E} = -\nabla\cdot(\nabla\Phi) = -\nabla^2\Phi = \frac{\rho}{\varepsilon_0}$ (Poisson's Eq.)	$\nabla\times\overline{H} = \overline{J}$

$$\Phi(x,y,z) = \iiint_{x',y',z'} \frac{\rho(x',y',z')\,dx'dy'dz'}{4\pi\varepsilon_0\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{\frac{1}{2}}}$$

$$\nabla\cdot\left(\mu_0\overline{H}\right) = 0 \Rightarrow \mu_0\overline{H} = \nabla\times\overline{A}$$

$$\nabla^2\overline{A} = -\mu_0\overline{J},\ \nabla\cdot\overline{A} = 0$$

$$\overline{A}(x,y,z) = \iiint_{x',y',z'} \frac{\mu_0\overline{J}(x',y',z')\,dx'dy'dz'}{4\pi\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{\frac{1}{2}}}$$