6.641, Electromagnetic Fields, Forces, and Motion
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Lecture 6: Magnetoquasistatics

I. MQS Governing Equations

$$\nabla \cdot \left(\mu_0 \overline{H}\right) = 0 \Rightarrow \mu_0 \overline{H} = \nabla \times \overline{A} \qquad \left(\overline{A} = \text{vector potential}\right)$$

$$\nabla \times \overline{H} = \frac{1}{\mu_0} \nabla \times \left(\nabla \times \overline{A}\right) = \overline{J}$$

$$\nabla \times \left(\nabla \times \overline{A}\right) = \nabla\left(\nabla \cdot \overline{A}\right) - \nabla^2 \overline{A} = \mu_0 \overline{J}$$

II. Uniqueness

If $\overline{A} \rightarrow \overline{A} + \nabla \chi$, $\nabla \times \overline{A}$ is unchanged because $\nabla \times \left(\nabla \chi\right) = 0$

For $\nabla \cdot \overline{A}$ to also remained unchanged requires $\nabla^2 \chi = 0$

When, for EQS systems

$$\nabla^2 \Phi = \frac{-\rho}{\varepsilon} \Rightarrow \Phi\left(\overline{r}\right) = \int_{V'} \frac{\rho\left(\overline{r}'\right) dV'}{4\pi\varepsilon \left|\overline{r} - \overline{r}'\right|}$$

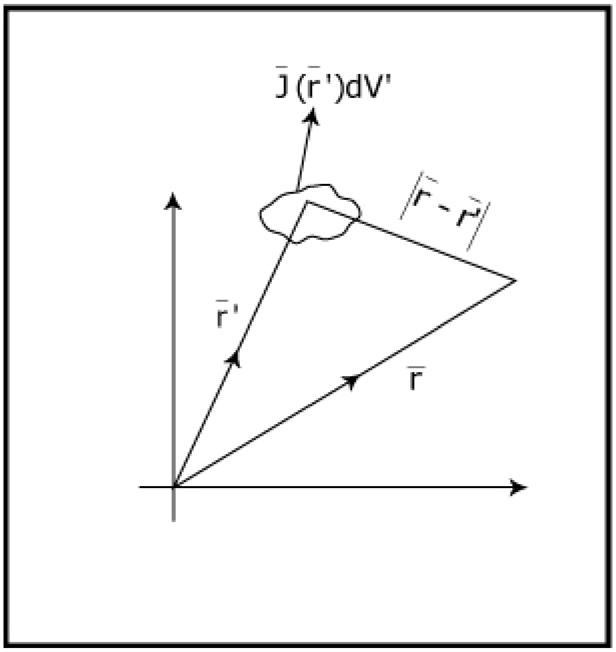
For $\nabla^2 \chi = 0$ everywhere, it is analogous to $\rho = 0$ everywhere for which $\Phi\left(\overline{r}\right) = 0$.

Thus to uniquely specify a vector to within a constant, both its curl and divergence must be specified. Here, we have thus far specified $\nabla \times \overline{A} = \mu_0 \overline{H}$. We are free to specify $\nabla \cdot \overline{A}$ to any convenient value. We choose $\nabla \cdot \overline{A} = 0$ which is called setting the gauge. Then

$$\nabla^2 \overline{A} = -\mu_0 \overline{J}$$

III. Vector Poisson's equation

$$\nabla^2 \overline{A} = -\mu_0 \overline{J} \Rightarrow \nabla^2 A_x = -\mu_0 J_x$$
$$\nabla^2 A_y = -\mu_0 J_y$$
$$\nabla^2 A_z = -\mu_0 J_z$$



(Important fact: For each current element $\overline{J}(\overline{r}')dV'$, the contribution to $\overline{A}$ is in the same direction as $\overline{J}$.)

In analogy to the EQS Poisson's equation

$$A_x(\overline{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{J_x(\overline{r}')\,dV'}{|\overline{r} - \overline{r}'|}$$

$$A_y(\overline{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{J_y(\overline{r}')\,dV'}{|\overline{r} - \overline{r}'|}$$

$$A_z(\overline{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{J_z(\overline{r}')\,dV'}{|\overline{r} - \overline{r}'|}$$

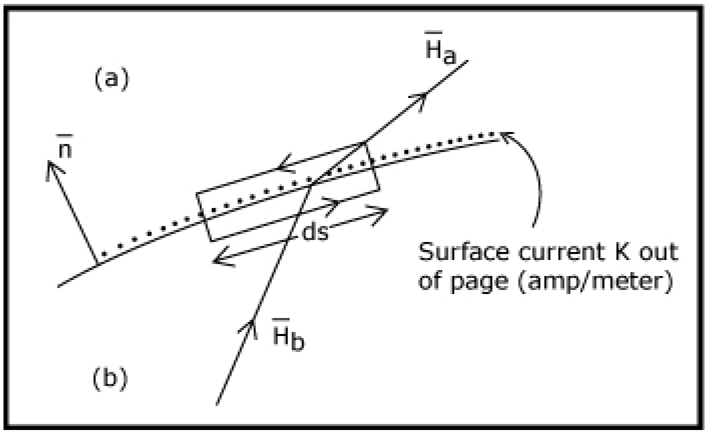
or in compact vector form

$$\overline{A}(\overline{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\overline{J}(\overline{r}')\,dV'}{|\overline{r} - \overline{r}'|}$$

IV. Boundary Conditions

1. Tangential $\overline{H}$

$$\nabla \times \overline{H} = \overline{J} \Rightarrow \oint_C \overline{H} \bullet \overline{ds} = \int_S \overline{J} \bullet \overline{da}$$

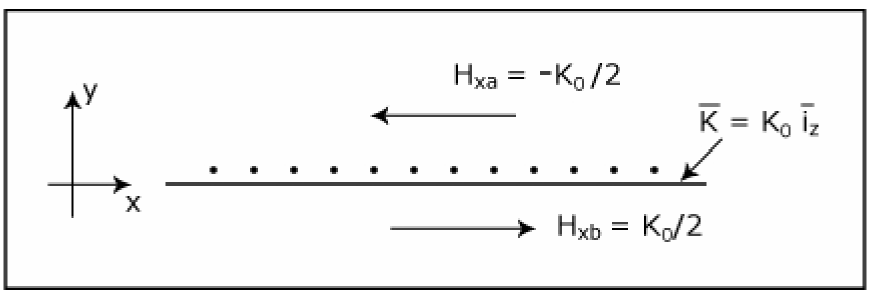


$$H_{bt}\cancel{ds} - H_{at}\cancel{ds} = K\cancel{ds}$$

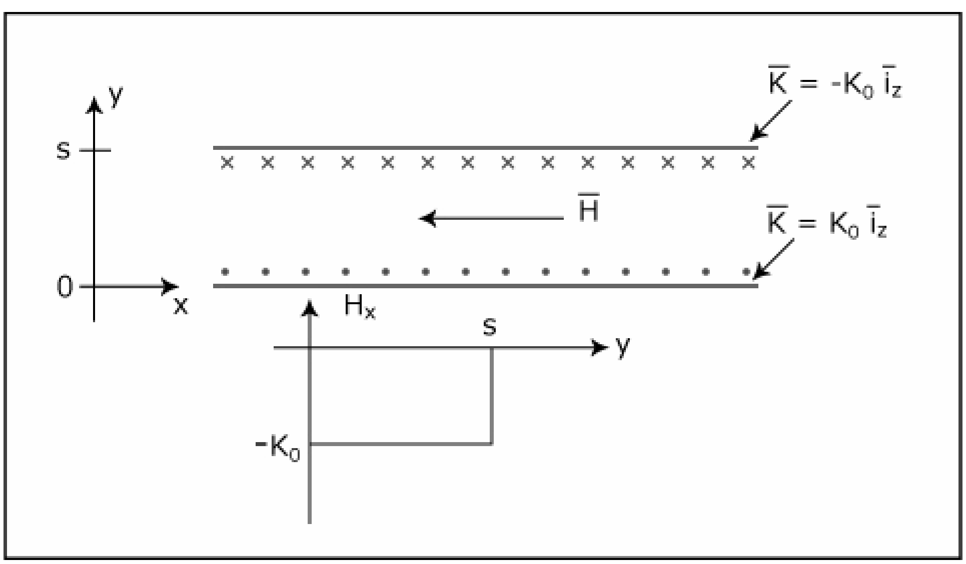
$$H_{bt} - H_{at} = K$$

$$\overline{n} \times \left[\overline{H}_a - \overline{H}_b\right] = \overline{K}$$

2. Single Current Sheet

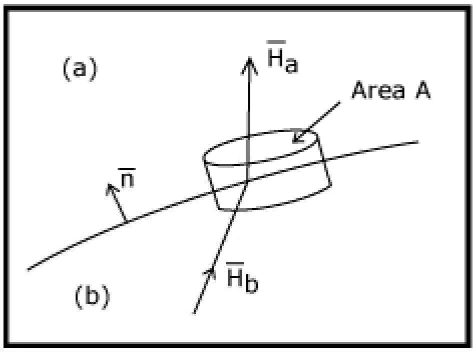


3. Two Oppositely Directed Current Sheets



4. Normal H

$$\nabla \cdot \mu_0 \overline{H} = 0 \Rightarrow \oint_S \mu_0 \overline{H} \cdot \overline{da} = 0$$



$$\mu_0 \left(H_{an} - H_{bn}\right) A = 0$$

$$H_{an} = H_{bn}$$

$$\overline{n} \cdot \left[\overline{H}_a - \overline{H}_b\right] = 0$$

V. Biot – Savart Superposition Integral

$$\overline{H} = \frac{1}{\mu_0} \nabla \times \overline{A} = \frac{1}{4\pi} \nabla \times \int_{V'} \frac{\overline{J}\left(\overline{r}'\right) dV'}{\left|\overline{r} - \overline{r}'\right|}$$

$$= \frac{1}{4\pi} \int_{V'} \nabla \times \left[\frac{\overline{J}\left(\overline{r}'\right) dV'}{\left|\overline{r} - \overline{r}'\right|}\right]$$

Let $\chi = \dfrac{1}{\left|\overline{r} - \overline{r}'\right|}$

$$\nabla \times \left(\chi \overline{J}\left(\overline{r}'\right)\right) = \chi \cancelto{0}{\nabla \times \left(\overline{J}\left(\overline{r}'\right)\right)} + \nabla \chi \times \overline{J}\left(\overline{r}'\right)$$

In a spherical coordinate system: $\nabla \left(\dfrac{1}{r}\right) = \overline{i}_r \dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\right) = -\dfrac{1}{r^2} \overline{i}_r$

Therefore: $\nabla \left(\dfrac{1}{\left|\overline{r} - \overline{r}'\right|}\right) = \dfrac{-\overline{i}_{r'r}}{\left|\overline{r} - \overline{r}'\right|^2}$

$$\overline{i}_{r'r} = \frac{\overline{r} - \overline{r}'}{\left|\overline{r} - \overline{r}'\right|}$$

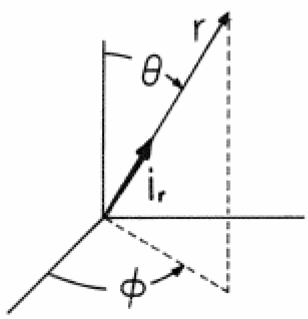


Figure 8.2.1 Spherical coordinate system with $\mathbf{r}'$ located at origin.

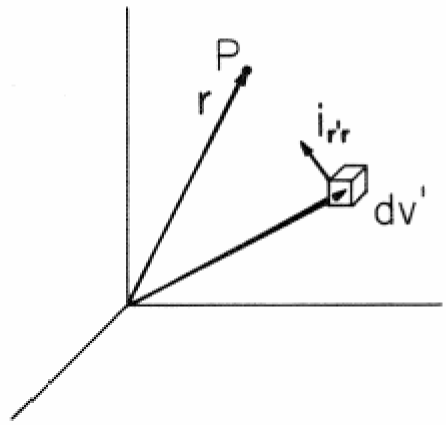


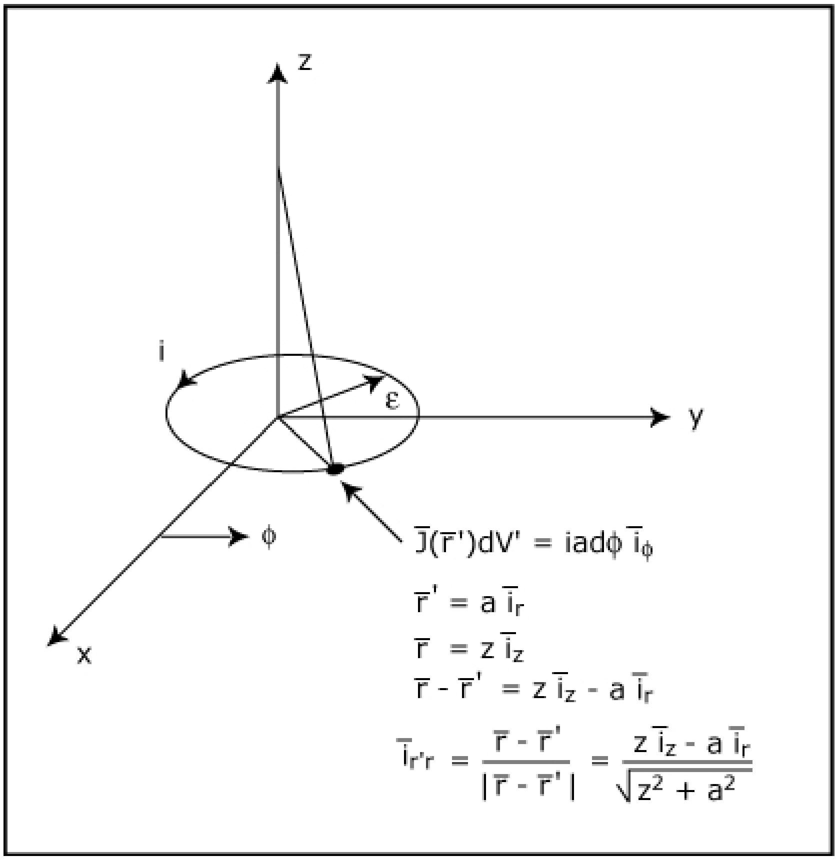
Figure 8.2.2 Source coordinate $\mathbf{r}'$ and observer coordinate $\mathbf{r}$ showing unit vector $\mathbf{i}_{r'r}$ directed from $\mathbf{r}'$ to $\mathbf{r}$.

$$\nabla \times \left(\frac{\bar{J}(\bar{r}\,')}{\left|\bar{r} - \bar{r}\,'\right|} \right) = \frac{-\,\bar{i}_{r'r}}{\left|\bar{r} - \bar{r}\,'\right|^2} \times \bar{J}(\bar{r}\,')$$

$$= \frac{\bar{J}(\bar{r}\,') \times \bar{i}_{r'r}}{\left|\bar{r} - \bar{r}\,'\right|^2}$$

$$\bar{H} = \frac{1}{4\pi} \int_{V'} \frac{\bar{J}(\bar{r}\,') \times \bar{i}_{r'r}\, dV'}{\left|\bar{r} - \bar{r}\,'\right|^2}$$

VI. On Axis Magnetic Field from Current Loop



$$\bar{J}(\bar{r}') \times \bar{i}_{r'r}\, dV' = iad\phi\, \bar{i}_\phi \times \left[\frac{z\,\bar{i}_z - a\,\bar{i}_r}{\sqrt{z^2 + a^2}}\right] = \frac{iad\phi}{\sqrt{z^2 + a^2}}\left[z\,\bar{i}_r + a\,\bar{i}_z\right]$$

$$\bar{H} = \frac{1}{4\pi}\int_{\phi=0}^{2\pi} \frac{ia\left[z\,\bar{i}_r + a\,\bar{i}_z\right]d\phi}{\sqrt{z^2 + a^2}\left[z^2 + a^2\right]} = \frac{ia^2\,\bar{i}_z}{2\left[z^2 + a^2\right]^{3/2}}$$

Hint: $\bar{i}_r = \cos\phi\,\bar{i}_x + \sin\phi\,\bar{i}_y \Rightarrow \int_0^{2\pi} \bar{i}_r\, d\phi = 0$

VII. On Axis Magnetic Field of Circular Cylindrical Solenoid

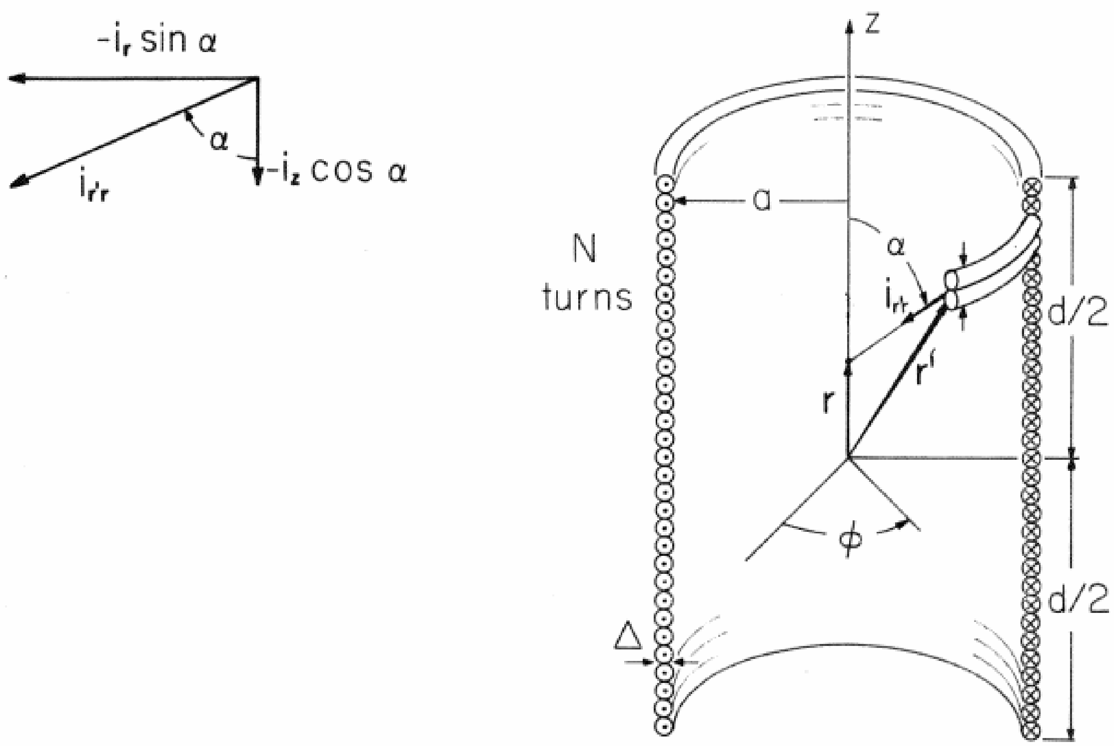


Figure 8.2.3 A solenoid consists of N turns uniformly wound over a length d, each turn carrying a current i. The field is calculated along the z axis, so the observer coordinate is at $\mathbf{r}$ on the z axis.

A. Superposition Approach Using Previous Result of Single Current Loop

Consider the solenoid as a collection of current loops, each of length dz'. For the loop at z' of thickness dz', the current in the loop is $di = \frac{Ni}{d} dz'$. The magnetic field from this loop is

$$d\overline{H} = \frac{di\, a^2}{2\left[(z-z')^2 + a^2\right]^{3/2}} \bar{i}_z = \frac{Ni\, a^2 dz'}{2d\left[(z-z')^2 + a^2\right]^{3/2}} \bar{i}_z$$

The total magnetic field is

$$H_z = \int_{z'=-\frac{d}{2}}^{+d/2} \frac{Ni\, a^2 dz'}{2d\left[(z-z')^2 + a^2\right]^{3/2}} = \frac{Ni\, a^2}{2d} \int_{z'=-\frac{d}{2}}^{+d/2} \frac{dz'}{\left[(z-z')^2 + a^2\right]^{3/2}}$$

$$= \frac{Nia^2}{2d} \left. \frac{(z'-z)}{a^2\left[a^2+(z-z')^2\right]^{1/2}} \right|_{z'=-\frac{d}{2}}^{+\frac{d}{2}}$$

$$= \frac{Ni}{2d}\left[\frac{\frac{d}{2}-z}{\left[a^2+\left(z-\frac{d}{2}\right)^2\right]^{1/2}} + \frac{\left(\frac{d}{2}+z\right)}{\left[a^2+\left(z+\frac{d}{2}\right)^2\right]^{1/2}}\right]$$

$$\lim_{\frac{d}{2}\gg z} H_z = \frac{Ni}{2\cancel{d}} \frac{\cancel{d}}{\left[a^2+\left(\frac{d}{2}\right)^2\right]^{1/2}}$$

$$\lim_{\frac{d}{2}\gg a} H_z = \frac{Ni}{d}$$

B. Solenoid modeled as Surface Current $\overline{K} = \frac{Ni}{d}\,\overline{i}_\phi$

$$\overline{H} = \frac{1}{4\pi}\int_{S'} \frac{\overline{K}\times\overline{i}_{r'r}\, dS}{\left|\overline{r}-\overline{r}'\right|^2}$$

$$\overline{i}_{r'r} = -\,\overline{i}_r \sin\alpha - \overline{i}_z \cos\alpha$$

$$\sin\alpha = \frac{a}{\sqrt{a^2+(z'-z)^2}},\ \cos\alpha = \frac{(z'-z)}{\sqrt{a^2+(z'-z)^2}},\ \left|\overline{r}-\overline{r}'\right|^2 = \left[a^2+(z'-z)^2\right]$$

$$\overline{H} = \frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{z'=-\frac{d}{2}}^{+d/2} \frac{Ni}{d}\frac{\overline{i}_\phi\times\left[-\,\overline{i}_r\sin\alpha - \overline{i}_z\cos\alpha\right] a\,d\phi\,dz'}{\left[a^2+(z'-z)^2\right]}$$

Note: $\int_{\phi=0}^{2\pi} \overline{i}_r\, d\phi = 0$

$$\bar{H} = \frac{Ni}{4\pi d} 2\pi a \int_{z'=-\frac{d}{2}}^{+d/2} \frac{\bar{i}_z \sin\alpha}{\left[a^2 + (z'-z)^2\right]} dz'$$

$$H_z = \frac{Nia^2}{2d} \int_{z'=-\frac{d}{2}}^{+d/2} \frac{dz'}{\left[a^2 + (z'-z)^2\right]^{3/2}} = \frac{Ni\cancel{a^2}}{2d} \left. \frac{(z'-z)}{\cancel{a^2}\left[a^2 + (z'-z)^2\right]^{1/2}} \right|_{z'=-\frac{d}{2}}^{+d/2}$$

$$= \frac{Ni}{2d} \left[\frac{\frac{d}{2} - z}{\left[a^2 + \left(z - \frac{d}{2}\right)^2\right]^{1/2}} + \frac{\frac{d}{2} + z}{\left[a^2 + \left(z + \frac{d}{2}\right)^2\right]^{1/2}} \right]$$

VIII. Demonstration 8.2.1 Fields of a Circular Cylindrical Solenoid

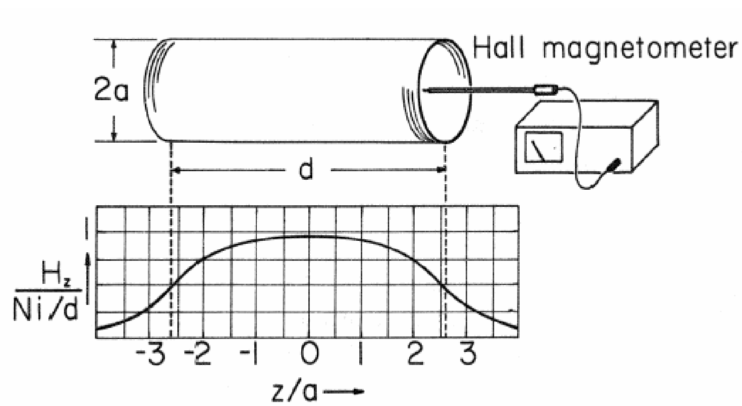


Figure 8.2.4 Experiment for documenting the axial **H** predicted in Example 8.2.1. Profile of normalized H_z is for $d/2a = 2.58$.