6.890	Lecture 3	Sept. 11, 2014

Recall: packing of n squares without rotation into a square is strongly NP-complete [L2]

Edge-unfolding polyhedra: given a polyhedron, cut along edges to unfold flat without overlap

- not always possible [Biedl et al. & Bern et al. 1998]
- strongly NP-hard [Abel & Demaine 2011]
  even for orthogonal polyhedra topologically sphere

Reduction from Square Packing:

- infrastructure: polyhedron with square with tower with squares & "atoms" on side
- "pipe" is super long but can move out

⇒ squares must pack inside base of tower

- atoms are universal: can turn left/right/straight in 2D unfolding & left/right/straight on tower surface

⇒ can connect & place squares as in any (slightly perturbed) packing, then exit via pipe

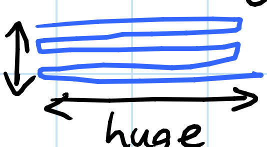
- lots of details e.g. shrink squares slightly to enable perturbation

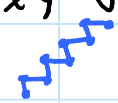
Snake cube puzzle: AKA Cubra circa 1990

- given chain of unit cubes each with specified "turn angle" of 0 or 90° (elastic through centers)
- goal: fold it into larger cube (exactly)

- NP-hard
  [Abel, Demaine, Demaine, Eisenstat, Lynch, Schardl 2012]

Reduction from 3-Partition:

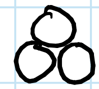
- infrastructure:
  - fill cube to leave $x \times y \times z$ box
  - fill box to leave "hub & slots" shape
  - each hub is $8 \times t \times$ huge
- a_i gadget: $8a_i$ 

  must go in 1 hub
  - 8 to avoid coming back to same $4 \times 4 \times 4$ voxel
- connected together by zig-zag gadget 
- zig-zag is universal:
  - $2 \times 2 \times 2$ can turn/go straight
    ⇒ fill Hamiltonian shapes scaled $2\times$
  - $2 \times 2 \times 2$ refinement makes any shape Hamilt.
  - ⇒ $4 \times 4 \times 4$ refinement makes fillable by zig-zag
  - parity issue: snake alternates in cell parity
  - claim: can start & end at any faces of opposite parity

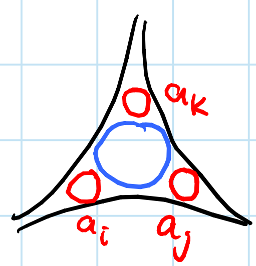
Disk packing: pack n given disks into given shape

- **motivation:** computational origami design
  (tree method — see Lang)
- strongly NP-hard [Demaine, Fekete, Lang — OSME 2010]

Reduction from 3-Partition:

- infrastructure:
  - build $n/3$ symmetric  pockets
  - equilateral Δ: forced packing
  - square target: forced packing
    + repeated subdivision with forced packings
    + fill all other pockets by repeatedly adding maximal disks, until small enough

    (depth $\approx \lg n$)
- triple gadget: (in symmetric pocket)
  - scale a_i's & t so that $t = 1$
  - shrink center disk by $-1/N$ (↳ big)
  - shrink a_i disk by $-1/N^2$,
    grow it by $+a_i/N$
  - key property: disks fit $\Leftrightarrow a_i + a_j + a_k \le t$
    (proof by geometry + Taylor series)



Clickomania: [Schuessler ~2000?]

- given rectangular grid of colored squares
- move = remove connected group of >1 square of the same color
  - remaining squares fall within each column
  - empty columns disappear

- polynomial for one row or column
  - reduces to CFG parsing

  $S \to \Lambda \mid SS \mid c_i S c_i \mid c_i S c_i S c_i$

- NP-hard for
  - 2 columns & 5 colors
  - 5 columns & 3 colors

  [Biedl, Demaine, Demaine, Fleischer, Jacobsen, Munro 2000]
- OPEN: 2 rows?
  2 colors?

Reduction from 3-Partition:

→ necessary: encoding in unary

- left column mostly checkerboard except middle & interspersed red □s to measure t's
  - collapses ⇔ red □s removed
- right column has a_i groups + red squares on top

  ← scaled by B $= \frac{4}{3}n$
- details: spacing out groups & reds while still getting alignment

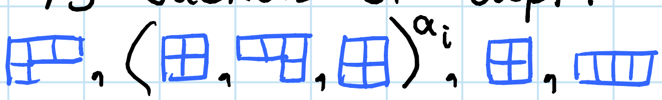
Tetris: [Alexey Pazhitnov 1985]

- retangular board
- tetromino blocks come one at a time
  - ↳ 4 unit squares joined edge-to-edge

- can rotate block as it falls from sky
- filled lines disappear
- stack to sky ⇒ die

- perfect information version:
  - know entire sequence of pieces to come
  - initial board position given
- NP-complete to [Breukelaar, Demaine, Hohenberger, Hoogeboom, Kosters, Liben-Nowell 2003]
  - survive
  - approximate # lines/Tetrises/time until death up to a factor of $n^{1-\varepsilon}$

Reduction from 3-Partition: → necessary: encoding in unary

- initial board = $n/3$ buckets of "depth" t
- a_i encoded as 
  - claim: entire gadget must go in one bucket
- finale = 

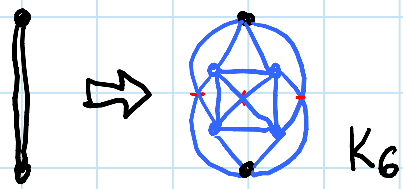
OPEN:
- initially empty board
- $O(1)$ rows or columns
- restricted piece sets (e.g. )
- no last-minute slides
- 2-player: PSPACE-complete?
- online Tetris?

1-planarity: draw a given graph in the plane such that each edge crosses ≤ 1 other [Ringel 1985]

- NP-complete [Grigoriev & Bodlaender — Alg. 2007]

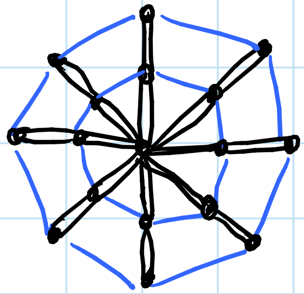
Reduction from 3-Partition:

- uncrossable edge gadget:

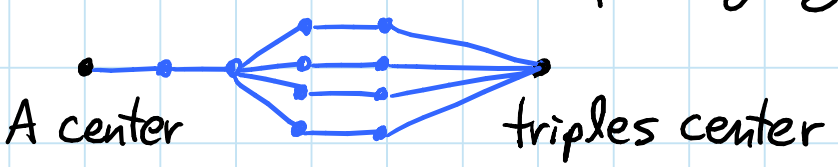


(denoted by thick edge)

- double wheel gadget:
  - unique embedding
  - one for A
  - one for triples
  - separate triples with thick edges every t hours around triples gadget



- a_i gadget:



GeoLoop & Ivan's Hinge puzzles: piano-hinged dissection
↳ NP-complete from 3-Partition

Ruler folding:
- given carpenter's ruler with lengths $a_1, a_2, \ldots, a_n$
- goal: fold to fit in 1D box of length L

- weakly NP-complete [Hopcroft, Joseph, Whitesides – SICOMP 1985]
- pseudopolynomial (like 2-Partition)

Reduction from (2-)Partition:
- idea: Partition solvable ⟺ can assign signs to a_i's such that $\sum_i \pm a_i = \emptyset$
- folding flips sign; unfolding leaves sign
⇒ can fold ends together ⟺ Partition solvable
- construction: $2B, B, a_1, a_2, \ldots, a_n, B, 2B$
↳ $> \sum_i a_i$

⇒ 2B's will be aligned & fit inside length-2B box
⟺ can fold ends together ⟺ Partition solvable

Map folding (simple): given crease pattern, can it fold flat by sequence of simple folds?

- weakly NP-hard [Arkin, Bender, Demaine, Demaine, Mitchell, Sethia, Skiena – 2000]

  for orthogonal paper & orthogonal creases
  or square paper & 45°/orthog. creases

Reduction from Partition:
- similar to Ruler Folding
- 2 vertical creases check y extent against frame
- horizontal creases done before or after check
  (before: if ruler folded; after: if not)

- force square paper into orthogonal shape:

OPEN: strongly NP-hard?
pseudopolynomial?

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