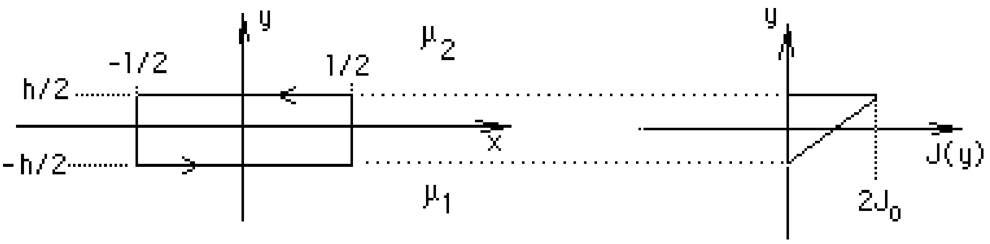
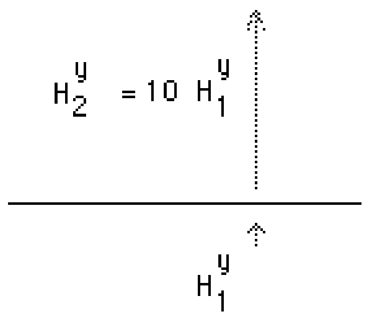
Solutions. Chapter 2

2.1 a)



b) For the normal component of $\boldsymbol{H}$, it is still possible to use the boundary condition derived for normal induction. It does not depend on surface current but only on permeability:

$$(\boldsymbol{B}_2 - \boldsymbol{B}_1)\cdot\boldsymbol{n} = 0, \quad H_1^y\mu_1 = H_2^y\mu_2, \quad H_2^y / H_1^y = \mu_1/\mu_2 = 10:$$

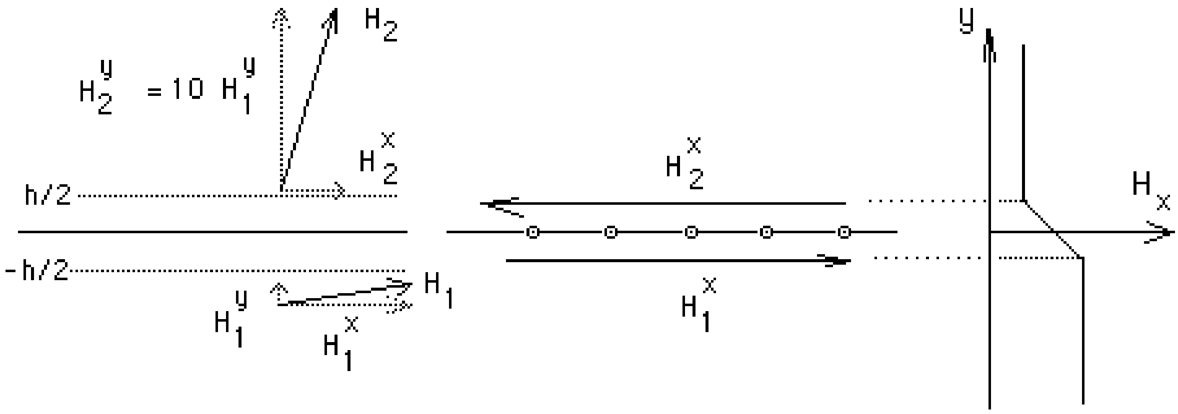


From Eq. 2.4, boundary conditions may be derived for H_x: $\int \boldsymbol{H}\cdot d\boldsymbol{l} = \int\int \boldsymbol{J}\cdot d\boldsymbol{A}$
Setting up these integrals gives (the length of the path is l, its height, h):

$$H_1 \sin\theta_1 - H_2 \sin\theta_2 = J_0\, h/2$$

where, for the LHS, the one goes counterclockwise about the path of integration.

How does H_x vary through the surface layer? It is necessary only to take different limits on the y integrations, enclosing less of the sheet current to see that H_x varies linearly from its value at -h/2 to that at h/2.
H_y on the other hand is discontinuous at $y = 0$
c) The x component of H_1 exceeds the x component of H_2 by the amount of J_0 h/2. Combining this with the above condition on the y component gives:



The decrease in H_x in medium 2 is consistent with the direction of the magnetic field produced by the surface current as shown above, center. That field augments H_x in medium 1 and tends to cancel H_x in medium 2. The variation of H_x through the current sheet is shown at right.

2.2 Want $B = \phi/A = 0.6$ T in the gap of the circuit.

a) How much current is required in the $N = 600$ turns on the 20 cm leg?
Assume the area over through which the flux passes in each leg of the circuit is the same. It is also sufficient to assume that the 600 turns act over the entire length of the circuit. Then

$$NI = \phi_m \sum R_m = \phi_m \sum \frac{l_i}{\mu_i A_i} = \frac{B}{\mu_0}\left(\frac{0.59}{100} + \frac{0.01}{1}\right) = \frac{0.6}{\mu_0} 1.59 \times 10^{-2}$$

In the first term of the sum $\mu_r = 100$. The second term is for the length of the circuit in the gap, $\mu_r = 1$. Here it is assumed the area over which the flux spreads in the gap, A_g, is the same as $A_i = 10^{-3}$ m^2. Thus the required current is

$$I = 12.7 \text{ A}$$

Notice that the larger term in the sum is that due to the gap. (If the magnetic circuit were 10 cm long rather than 60, the gap would make up 90% of the reluctance and for a 0.6 T gap field, $NI = (0.011)*0.6/\mu_o$. In this case, $I = 8$ A.)

b) The flux density in the iron core is given by solving the above NI equation for $(\phi/A)_{Fe}$. Because we have assumed the same area throughout the circuit, the flux density in the iron would also be 0.6 T. If we were to account for the fact that in the gap the flux spreads out over a larger area, then the flux density in the iron would have to be greater than 0.6T to maintain 0.6 T in the gap. Alternatively if you calculated the H field created by the coil $H = NI/l = 12.9$ kA/m. The demagnetizing factor reduces the effective permeability to about 60 (from 100). Then inside the iron $B = \mu_r' \mu_0 H = 0.97I$. This large value reflects the fact that to get 0.6 T in the gap, B in the iron must be higher due to flux spreading in the gap.

2.3 From Eq. 2.51 and using $\cos\theta = h/r = h/(h^2 + x^2)^{1/2}$, $\sin\theta = x/r = x/(h^2 + x^2)^{1/2}$, where x is the distance along the copper sheet, it follows

$$H = \frac{\mu_m}{(h^2 + x^2)^2}[2h\boldsymbol{e}_r + x\boldsymbol{e}_\theta]$$

The field lines shear to the right above and below the Cu sheet as you get closer to it. This is caused by the addition of the field due to the current and the field due to the dipole. The tension in these flux lines produces a force on the sheet to the left.

2.4 Start with a sample of soft ferromagnetic material at $z = -\infty$ and a permanent magnet at the origin (see Fig. below). Bring the soft magnet toward the origin (step 1 in figure) and two things happen as it moves into the dipole field given by Eq. 2.51: first the sample responds to the field it sees by an induced magnetization proportional to the field $M = \chi H$ (the field does work in magnetizing the sample) and second the sample experiences an attractive force due to its moment in the field gradient. The sum of these two energies is negative, but not as negative as it would be had the material already been magnetized. The force per unit volume experienced by the sample of volume V during this process is given by

$$f = F/V = M(\partial B/\partial x)$$

where M is the varying magnetization response of the sample to the dipole field. The work per unit volume done by this force in moving the sample to x_1 is given by

$$A_2 = \int_{\infty}^{x_1} M \frac{\partial B}{\partial x} dx = \int_0^{B_1} M dB$$

which is the expression given earlier for A_2. Note that A_2 includes the effect of magnetizing the sample.

The magnetized sample in the field has energy $-\boldsymbol{M}\cdot\boldsymbol{B}$.

Once this magnetization process is completed, if the magnitude of M is fixed at M_1 and the material is removed from the field (process 2, Fig. above), work must be done on the sample to remove it from the field:

$$W = \int_{B_1}^{0} M_1 dB = -M_1 B_1 = A_3$$

Thus A_3 measures the energy of a fixed magnetization density in a field. The minus sign implies work is done on the sample in removing it from the field. We now have a saturated sample at $z = -\infty$. The difference in energy between these two processes is the energy needed to magnetize the sample, $A_1 = |A_3| - A_2$. That is, the difference between bringing an unmagnetized sample from $B = 0$ to $B = B_1$, so that it has magnetization M_1, and then holding M_1 fixed versus removing the sample to $B = 0$, is simply the energy needed to assemble the randomly oriented atomic moments ($M = 0$) in the configuration that has $M = M_1$, regardless of the field. This energy density A_1 includes magnetic anisotropy and shape energy densities. A_1 is the energy of most importance to us. It is the energy needed to magnetize a sample, the anisotropy energy.

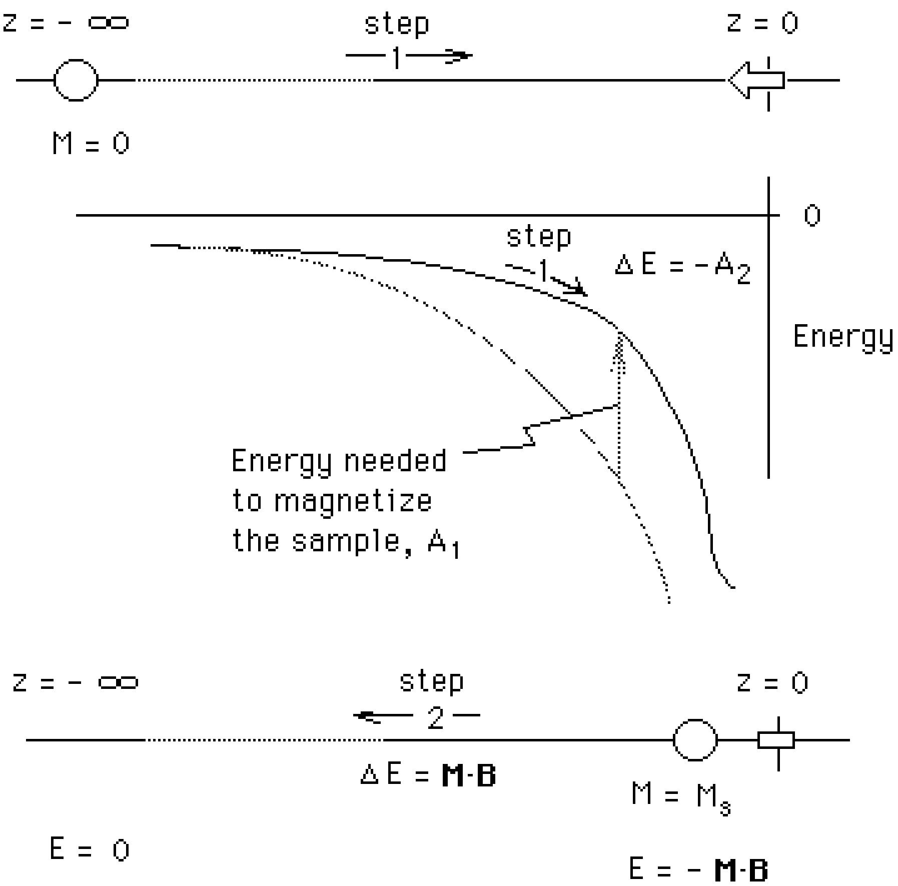


Fig. Gedanken experiment in which a demagnetized sample is first brought from negative infinity into the field of a permanent magnet at the origin. During this process the sample becomes magnetized and its energy is lowered by its attractive interaction with the dipole field. During this first process the sample's energy decreases by A_2. In the second process, the magnetization is held fixed and the sample is removed from the field. During this process it gains energy ***M.B***. The difference between these two processes is the energy needed to magnetize the sample $A_1 =$ ***M.B*** **-** $|A_2|$.

2.7 a) The remanent inductions are about 2 kG and 0.5 kG for the toroid and rod, respectively.

b) The coercivity is about 1.0 Oe in each case.

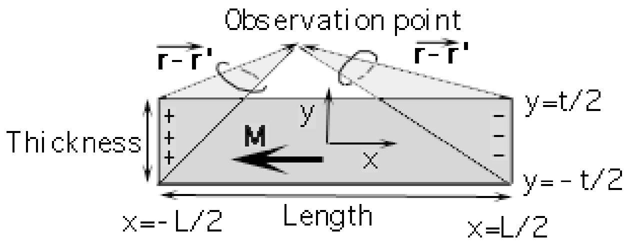
c) The effective permeability for a drive field of 2 Oe is about 2400/2 = 1200 for the toroid and 1000/2 or 500 for the rod. [If you convert *B* and *H* to *T* and *A/m* for MKS, then $\mu_r = B/(\mu_0 H)$]

d) The permeability is related to the susceptibility by $\mu = 1 + 4\pi\chi$, so $\chi = 96$ and from part c) $\chi_{eff} = 40$ for the rod. The demagnetization factor, *N*, for the rod is given from Eq. 2.29 as $N = 1/\chi_{eff} - 1/\chi = 0.015$.

e) Table 2.1 gives $N = 0.017$ in fair agreement with the calculated value in d).

2.8 solution, and then some. Field from *surface* poles (second part of Eq. 2.45) is given by the 3-*D* equation:

$$\boldsymbol{H}(\boldsymbol{r}) = \frac{1}{4\pi}\iint \boldsymbol{M}\cdot\boldsymbol{n}\,\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,d^2r'\,.$$



The problem asks for the x component of $\boldsymbol{H}$. In that case, $|\boldsymbol{r}-\boldsymbol{r}'|_x = (x + L/2)$ from the left surface where $r_1 = [(x + L/2)^2 + (y - y')^2 + (z - z')^2]^{1/2}$, and , $|\boldsymbol{r}-\boldsymbol{r}'|_x = (x - L/2)$ from the right surface where $r_2 = [(x - L/2)^2 + (y - y')^2 + (z - z')^2]^{1/2}$, and $\boldsymbol{M}\cdot\boldsymbol{n} = -M$. Thus:

$$H_x(x,y) =$$

$$\frac{M}{4\pi}\int_{-t/2}^{t/2} dy' \int_{-w/2}^{w/2} dz' \left\{ \frac{x + L/2}{\left[(x + L/2)^2 + (y - y')^2 + (z - z')^2\right]^{3/2}} - \frac{x - L/2}{\left[(x - L/2)^2 + (y - y')^2 + (z - z')^2\right]^{3/2}} \right\}$$

The minus sign on the second term above comes from the negative value of $\boldsymbol{M}\cdot\boldsymbol{n}$ at $x = L/2$. Note that the contributions to the field from *both* surfaces are positive because $x - L/2 \le 0$.

To do the dz' integral, let $\xi = z - z'$, $d\xi = -dz'$, so the limits of integration, $z' = w/2$ and $z' = -w/2$ become $\xi = z - w/2$ and $\xi = z + w/2$, respectively, giving:

3-*D*

$$H_x(x,y) = \frac{M}{4\pi}\int_{-t/2}^{t/2} dy' \int_{z-w/2}^{z+w/2} d\xi \left\{ \frac{x + L/2}{(a^2 + \xi^2)^{\frac{3}{2}}} - \frac{x - L/2}{(b^2 + \xi^2)^{\frac{3}{2}}} \right\}.$$

Here, for brevity, we have temporarily set $a^2 = (x + L/2)^2 + (y - y')^2$ and $b^2 = (x - L/2)^2 + (y - y')^2$. (Note the change of $-d\xi$ to $d\xi$ and reversal of the limits of integration over $d\xi$.)

Use the integral $\int \frac{dx}{(a^2 + x^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{a^2 + x^2}}$ to get:

3-*D*

$$H_x(x,y) = \frac{M}{4\pi}\int_{-t/2}^{t/2} dy' \left[\frac{\xi(x + L/2)}{a^2\sqrt{a^2 + \xi^2}} - \frac{\xi(x - 1/2)}{b^2\sqrt{b^2 + \xi^2}} \right]_{z-w/2}^{z+w/2}$$

$$= \frac{M}{4\pi}\int_{-t/2}^{t/2} dy' \left\{ \frac{(z + \frac{w}{2})(x + \frac{L}{2})}{a^2\sqrt{a^2 + (z + \frac{w}{2})^2}} - \frac{(z - \frac{w}{2})(x + \frac{L}{2})}{a^2\sqrt{a^2 + (z - \frac{w}{2})^2}} - \frac{(z + \frac{w}{2})(x - \frac{L}{2})}{b^2\sqrt{b^2 + (z + \frac{w}{2})^2}} + \frac{(z - \frac{w}{2})(x - \frac{L}{2})}{b^2\sqrt{b^2 + (z - \frac{w}{2})^2}} \right\}$$

Along symmetry plane $z = 0$ (for comparison with 2-D solution, which by definition is independent of z) this becomes:

3-*D*

$$H_x(x,y) = \frac{M}{4\pi}\int_{-t/2}^{t/2} dy' \left\{ \frac{w(x+L/2)}{a^2\sqrt{a^2+(w/2)^2}} - \frac{w(x-L/2)}{b^2\sqrt{b^2+(w/2)^2}} \right\}.$$

We will come back to this integral below, but for now we go to the 2-D limit by letting w approach infinity compared to x and y variables and limits. Thus, for fields that vary only in 2-D, (x and y) we have:

2-*D*:

$$H_x(x,y) = \frac{M}{2\pi}\int_{-t/2}^{t/2} dy' \left\{ \frac{(x+L/2)}{(x+L/2)^2+(y-y')^2} - \frac{(x-L/2)}{(x-L/2)^2+(y-y')^2} \right\}.$$

Make the substitution $y - y' \rightarrow \xi$ so $dy' \rightarrow -d\xi$. The limits of integration, $y' = \pm t/2$ are now: ξ - $t/2$ and ξ + $t/2$, respectively. As before, exchange integration limits and let $-d\xi \rightarrow d\xi$ to give:

2-D:

$$H_x(x,y) = \frac{M}{2\pi}\int_{y-\frac{t}{2}}^{y+\frac{t}{2}} \left[\frac{x+L/2}{(x+L/2)^2+\xi^2} - \frac{x-L/2}{(x-L/2)^2+\xi^2} \right] d\xi .$$

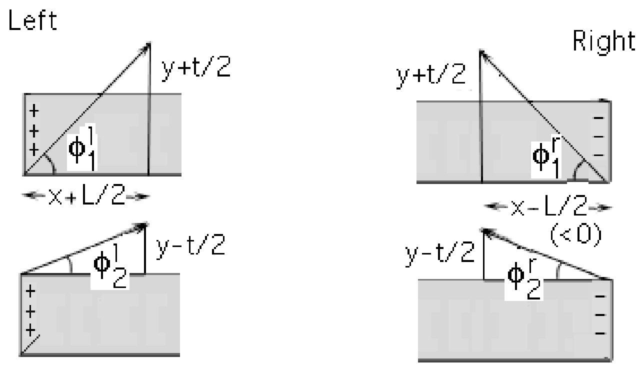
To do the integration, use $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan(\frac{x}{a})$ giving:

2-D

$$H_x(x,y) = \frac{M}{2\pi}\left[\arctan(\frac{\xi}{x+L/2}) - \arctan(\frac{\xi}{x-L/2}) \right]_{y-t/2}^{y+t/2}$$

$$= \frac{M}{2\pi}\left[\tan^{-1}\left(\frac{y+t/2}{x+L/2}\right) - \tan^{-1}\left(\frac{y-t/2}{x+L/2}\right) - \tan^{-1}\left(\frac{y+t/2}{x-L/2}\right) + \tan^{-1}\left(\frac{y-t/2}{x-L/2}\right) \right]$$

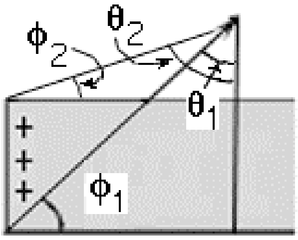
These four arctangents define four angles, ϕ_i, between the horizontal and the vector to the upper and lower limits of the linear charge distributions at $x = \pm L/2$.

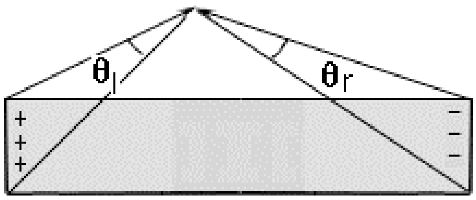


The angles ϕ_i convert to angles relative to the vertical, $\theta_i = \pi - \phi_i$ so $\phi_1 - \phi_2 = \pi - \theta_1 - (\pi - \theta_2) = \theta_2 - \theta_1$. Taking account of the fact that $x - L/2 \le 0$, we have:

$$H_x(x,y) = \frac{M}{2\pi}(\underbrace{\theta_2^{\text{left}} - \theta_1^{\text{left}}}_{\text{left:+poles}} + \underbrace{\theta_2^{\text{right}} - \theta_1^{\text{right}}}_{\text{right:−poles}}).$$

The differences in these θ_s define the angles subtended by the charge distribution from the observation point.

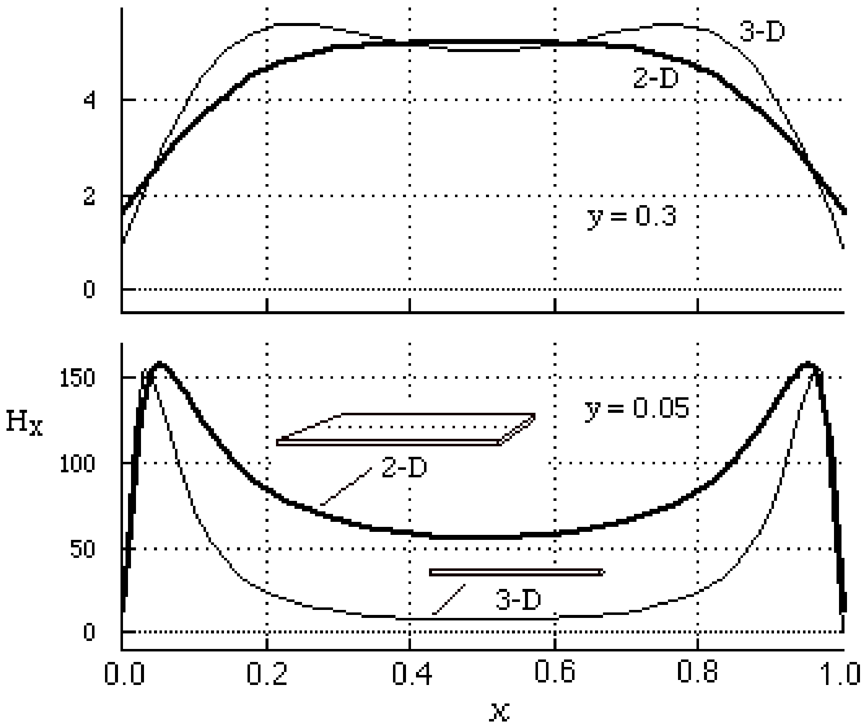




So finally

2-D $$H_x(x,y) = \frac{M}{2\pi}(\theta_{\text{left}} + \theta_{\text{right}}),$$

which is Eq. 2.3 for two surfaces having the polarity shown here. We now apply this equation to problem 2.8, where $t/2 = L/400$.



The results are displayed on an arbitrary vertical scale because obviously the 3D fields drop off much more strongly with y. The important point to get is that near the center of the magnet, and close to its surface, the field is very weak. This fact was used to calculate the field *inside* a solenoid (Ch. 1).

Let us now return to the last 3-D equation above,

3-D
$$H_x(x,y) = \frac{M}{4\pi}\int_{-t/2}^{t/2} dy'\left\{\frac{w(x+L/2)}{a^2\sqrt{a^2+(w/2)^2}} - \frac{w(x-L/2)}{b^2\sqrt{b^2+(w/2)^2}}\right\},$$

and consider the limit in which the strip is not very wide compared to x and $L/2$. For small w (to contrast wit the purely 2-D case) this becomes:

3-D
$$H_x(x,y) = \frac{Mw}{4\pi}\int_{-t/2}^{t/2} dy'\left\{\frac{x+L/2}{a^3} - \frac{x-L/2}{b^3}\right\}$$

where, as before, $a^2 = (x^2 + L/2)^2 + (y - y^1)^2$ and $b^2 = (x - L/2)^2 + (y - y')^2$. Setting $y - y' = \xi$ as before gives:

3-D
$$H_x(x,y) = \frac{Mw}{4\pi}\int_{y-\frac{t}{2}}^{y+\frac{t}{2}} d\xi\left\{\frac{x+L/2}{[(x+L/2)^2+\xi^2]^{\frac{3}{2}}} - \frac{x-L/2}{[(x-L/2)^2+\xi^2]^{\frac{3}{2}}}\right\}.$$

This form of integral was solved above, and now gives

3-D
$$H_x(x,y) = \frac{Mw}{4\pi}\left[\frac{\xi(x+L/2)}{(x+L/2)^2\sqrt{(x+L/2)^2+\xi^2}} - \frac{\xi(x-L/2)}{(x-L/2)^2+\xi^2}\right]_{y-t/2}^{y+t/2}.$$

$$= \frac{Mw}{4\pi}\left\{\frac{y+t/2}{(x+L/2)\sqrt{(x+L/2)^2+(y+t/2)^2}} - \frac{y-t/2}{(x+L/2)\sqrt{(x+L/2)^2+(y-t/2)^2}}\right.$$
$$\left. - \frac{y+t/2}{(x-L/2)\sqrt{(x-L/2)^2+(y+t/2)^2}} + \frac{y-t/2}{(x-L/2)\sqrt{(x-L/2)^2+(y-t/2)^2}}\right\}$$

These four terms define tangents as shown below.

Thus, for 3-*D*

$$H_x(x,y) = \frac{Mw}{4\pi}\left\{\frac{\tan\theta_2^{\text{left}}}{r_2^{\text{left}}} - \frac{\tan\theta_1^{\text{left}}}{r_1^{\text{left}}} - \frac{\tan\theta_2^{\text{right}}}{r_2^{\text{right}}} + \frac{\tan\theta_1^{\text{right}}}{r_1^{\text{right}}}\right\}$$

where $r_1^{\text{left}} = [(x+L/2)^2+(y+t/2)^2]^{1/2}$ etc. That is, the 3-*D* field (narrow *w*) along $z = 0$ is similar to the 2-*D* form but with angles replaced by angle tangents and there is an extra factor of w/2*r* in 3-*D*; the fields drop off faster. The 3-*D* form is plotted over the 2-*D* result for the two specified values of *y*.

In cylindrical coordinates, for the charged end faces of radius r_o, the field in 3-*D* is

$$H_x(r,x) = Mr_o\left[\frac{1}{x\sqrt{x^2+r}} + \frac{1}{(l-x)\sqrt{x^2+r^2}}\right]$$

which again scales like the ratio of the charge dimension over the distance to the observation point, r_0 / r.