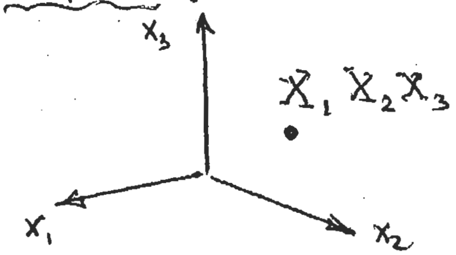
The Method of Direct Inspection

A useful algorithm exists through which one can deduce the value of a transformed tensor element by inspection, a method called (appropriately enough) the Method of Direct Inspection



Let's consider a point (could be the terminal point of a vector with coordinates $X_1\, X_2\, X_3$ (we use upper case symbols to distinguish coordinates from the labels attached to our reference axes)

Let's now examine the value of the product of a pair of these coordinates $X_i X_j$. If we change reference axes to a new set of basis vectors $x_1'\, x_2'\, x_3'$, the value of the product will change. Let's evaluate the new value of the product $X_i' X_j'$ (C'mon, you protest! I don't even see why you took the product in the first place!! Why should I care what the new value might be ?? Hold on! All will be revealed directly.)

The new value of the product $X_i' X_j'$ will be

$$X_i' X_j' = (C_{i\ell} X_\ell)\cdot(C_{jm} X_m)$$
$$= C_{i\ell} C_{jm} X_\ell X_m \quad \text{- - - - So what ??}$$

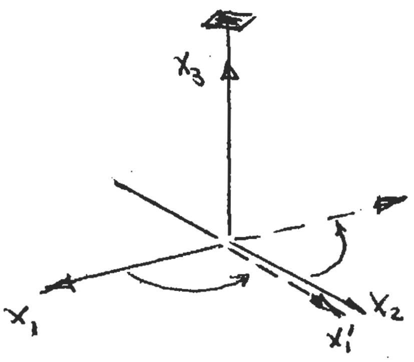
Well, the law for transformation for a tensor of second rank is

$$a_{ij}' = C_{i\ell} C_{jm} a_{\ell m}$$

which is analagous to the way in which the product of coordinates transforms (It is NOT identical!! Interchanging $C_{i\ell}$ and C_{jm} does not change the value of the product of the coordinates — but doing so for a_{ij}' gives a completely different expression!)

Therefore the elements of a tensor transform, upon a change of reference axes, in exactly the same way as the product of corresponding coordinates provided we maintain the correct order of terms

LET'S PROVIDE AN EXAMPLE AS THIS COOKBOOK RECIPE IS DECEPTIVELY SIMPLE. LET'S CONSIDER (ANTICIPATING AN APPLICATION OF THIS FORMALISM THAT IS SOON TO COME) A 4-FOLD ROTATION OF 90° ABOUT X_3



$$\begin{cases} x_1' = x_2 \\ x_2' = -x_1 \\ x_3' = x_3 \end{cases}$$

IF WE WISH TO DETERMINE THE NEW VALUE FOR, SAY, a_{12}' AFTER THIS CHANGE OF AXES WE EVALUATE

$$x_1' x_2' = x_2 (-x_1)$$

IF THE TENSOR ELEMENTS TRANSFORM LIKE THE PRODUCT OF COORDINATES, WE CAN SAY BY INSPECTION THAT

$$a_{12}' = -a_{21}$$

THAT IS, UPON THIS CHANGE OF AXES, THE NUMBER THAT APPEARS IN THE 1-2 BOX OF THE NEW TENSOR a_{ij}' IS THE NEGATIVE OF THE VALUE THAT APPEARED IN THE 2-1 BOX OF a_{ij}

THE METHOD IS DIRECTLY APPLICABLE TO TENSORS OF HIGHER RANK

IF WE WISH TO DETERMINE THE NEW VALUE OF, SAY, THE ELASTIC COMPLIANCE ELEMENT S_{1213} WE WOULD EXAMINE THE TRANSFORMATION OF THE PRODUCT $X_1 X_2 X_1 X_3$

FOR THE ABOVE CHANGE OF REFERENCE AXES

$$x_1' x_2' x_1' x_3' \longrightarrow x_2 (-x_1) x_2 x_3$$

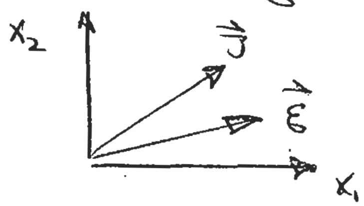
THE VALUE OF S_{1213}' IS THUS $-S_{2123}$

[WOULD YOU BELIEVE THAT WE HAVE JUST PERFORMED A QUADRUPLE SUMMATION THAT CONTAINED 81 TERMS!!??]

The Value of a Second-Rank Tensor Property in a Given Direction

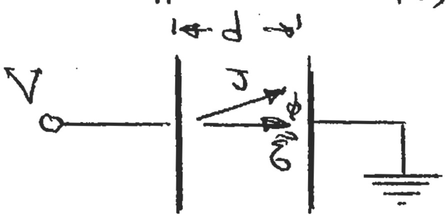
Let's consider a property such as electrical conductivity, a second-rank tensor that relates a current density vector, $\vec{J}$, to an applied electric field J_i (charge/unit area/unit time) $= \sigma_{ij}\ \mathcal{E}_j$ (volts/unit length)

The experiment we perform might be to subject a crystal to the electric field $\vec{\mathcal{E}}$ and then determine $\vec{J}$. $\vec{J}$ will NOT, in general, be parallel to $\vec{\mathcal{E}}$



When we measure σ (a scalar quantity!) then, to what "direction" does it pertain? The direction of $\vec{J}$? The direction of $\vec{\mathcal{E}}$? or should we split the difference and say the "direction" is midway between the two vectors?

Well, let's consider in detail what we might do in an actual experiment. Say we want to measure σ along the [111] direction. What we would do is to cut a plate normal to [111], attach electrodes and apply a voltage, V, providing a field, $\mathcal{E} = V/d$



The field $\vec{\mathcal{E}}$ is directed between the electrodes — ie, along [111] so clearly $\sigma_{[111]}$ will be the value of the property in the direction in which $\vec{\mathcal{E}}$, the "generalized force", is applied.

What, then, is the value of σ? $|J|/|\mathcal{E}|$ Right? Wrong! What we measure as the flux of charge parallel to $\vec{\mathcal{E}}$ as we measure the charge/unit area per unit time that passes between electrodes

$$\text{Thus } \sigma = J_{\parallel}/|\mathcal{E}| = \frac{|\vec{J}|\cos\phi}{|\mathcal{E}|} = \frac{\vec{J}\cdot\vec{\mathcal{E}}/|\mathcal{E}|}{|\mathcal{E}|}$$

$$\sigma = \frac{\vec{J}\cdot\vec{\mathcal{E}}}{|\mathcal{E}|^2}$$

WE CAN NOW WRITE THIS EXPRESSION IN TERMS OF THE CONDUCTIVITY TENSOR

$$\sigma = \frac{\vec{J} \cdot \vec{\mathcal{E}}}{|\mathcal{E}|^2} = \frac{J_i \mathcal{E}_i}{|\mathcal{E}|^2}$$

AS $\vec{J} \cdot \vec{\mathcal{E}} = J_1\mathcal{E}_1 + J_2\mathcal{E}_2 + J_3\mathcal{E}_3$

$$= \frac{\sigma_{ij}\, \mathcal{E}_j\, \mathcal{E}_i}{|\mathcal{E}|^2}$$

$\mathcal{E}_i$ IS $|\mathcal{E}|\, l_i$ WHERE $l_1\ l_2\ l_3$ ARE THE DIRECTION COSINES OF THE ORIENTATION IN WHICH WE APPLY $\mathcal{E}$

$$\sigma = \frac{\sigma_{ij}\, l_i l_j\, |\mathcal{E}||\mathcal{E}|}{|\mathcal{E}|^2}$$

SO THE VALUE OF σ IN A DIRECTION SPECIFIED BY DIRECTION COSINES l_i IS

$$\boxed{\sigma = \sigma_{ij}\, l_i l_j}$$

$$= \sigma_{11} l_1^2 + \sigma_{22} l_2^2 + \sigma_{33} l_3^2 + (\sigma_{12} + \sigma_{21}) l_1 l_2 + (\sigma_{13} + \sigma_{31}) l_1 l_3 + (\sigma_{23} + \sigma_{32}) l_2 l_3$$

WE CAN IMMEDIATELY USE THIS EXPRESSION TO PROVIDE A PHYSICAL INTERPRETATION OF SOME OF THE ELEMENTS IN A SECOND-RANK TENSOR.

LET'S ASK THE VALUE OF σ THAT WOULD BE MEASURED ALONG X_1

THE DIRECTION COSINES OF X_1 ARE $1, 0, 0$

$$\therefore\ \sigma = \sigma_{11}\, 1^2 + \sigma_{22}\, 0^2 + \sigma_{33}\, 0^2 + (\sigma_{12} + \sigma_{21})\, 1 \cdot 0 + (\sigma_{13} + \sigma_{31})\, 1 \cdot 0 + (\sigma_{23} + \sigma_{32})\, 0 \cdot 0$$

$$\boxed{\sigma = \sigma_{11}}$$

THUS THE DIAGONAL ELEMENTS σ_{ii} IN A SECOND RANK TENSOR REPRESENT THE VALUES OF THE PROPERTY THAT ONE WOULD MEASURE ALONG X_1, X_2 OR X_3, RESPECTIVELY.

"SYMMETRIC" TENSORS (AN UNFORTUNATE TERM)

A TENSOR σ_{ij} IS SAID TO BE "SYMMETRIC" IF $\sigma_{ij} \equiv \sigma_{ji}$

THIS EQUALITY COMES FROM ENERGY AND THERMODYNAMIC ARGUMENTS (AND NOT FROM CRYSTALLOGRAPHIC SYMMETRY!) THE EQUALITY MUST BE ESTABLISHED PROPERTY BY PROPERTY. MOST SECOND RANK PROPERTY TENSORS ARE SYMMETRIC, BUT NOT ALL! (THERMOELECTRICITY IS ONE EXAMPLE)

THE REPRESENTATION QUADRIC

WE CAN USE THE ELEMENTS OF A SECOND-RANK TENSOR TO DEFINE A SURFACE

$$a_{ij} x_i x_j = 1$$

THE SURFACE IS A QUADRATIC FORM AND CAN REPRESENT EITHER

AN ELLIPSOID

AN HYPERBOLOID OF ONE SHEET

AN HYPERBOLOID OF TWO SHEETS

AN IMAGINARY ELLIPSOID

THE EQUATION IS CALLED A "REPRESENTATION" SURFACE BECAUSE IT CONTAINS JUST ABOUT EVERYTHING ONE MIGHT LIKE TO KNOW ABOUT A SECOND RANK TENSOR. IN A PARTICULAR DIRECTION SPECIFIED BY DIRECTION COSINES l_i A RADIUS VECTOR, $\vec{R}$, FROM THE ORIGIN OUT TO THE SURFACE OF THE QUADRIC TERMINATES AT A POINT

$$x_1 = |R| l_1$$
$$x_2 = |R| l_2$$
$$x_3 = |R| l_3$$

THIS POINT SATISFIED THE EQUATION OF THE QUADRIC, SO

$$a_{ij} |R| l_i |R| l_j = 1$$

$$|R|^2 = \frac{1}{a_{ij} l_i l_j}$$

BUT THE DENOMINATOR IN THE TERM AT RIGHT IS THE VALUE OF THE PROPERTY, a, ALONG THE DIRECTION SPECIFIED BY l_1, l_2, l_3

THEREFORE THE RADIUS IN A GIVEN DIRECTION OF THE QUADRIC IS THE RECIPROCAL OF THE SQUARE ROOT OF THE PROPERTY IN THAT DIRECTION $\boxed{|R| = 1/\sqrt{a}}$ OR, CONVERSELY, $\boxed{a = \frac{1}{|R|^2}}$

THEREFORE, THE VARIATION OF A SECOND-RANK TENSOR PROPERTY VARIES SMOOTHLY WITH DIRECTION, WITH NO LUMPS, DIMPLES OR LOBES EVEN THOUGH SUCH VARIATION MIGHT BE COMPATIBLE WITH CRYSTAL SYMMETRY!