TENSOR PROPERTIES OF CRYSTALS AND ANISOTROPY

WHEN WE SPEAK OF A "PROPERTY" OF A MATERIAL WE MEAN — MAKING A DEFINITION IN FORMAL TERMS — THE RESPONSE OF A MATERIAL TO A SPECIFIC CHANGE IN A GIVEN SET OF CONDITIONS THAT RELATES INDEPENDENT AND DEPENDENT QUANTITIES IN A PARTICULAR PROCESS

WE SOMETIMES COLLECT PROPERTIES INTO SPECIFIC CATEGORIES

EG
- MECHANICAL
- ELECTRICAL

OTHER PROPERTIES ARE NOT QUITE SO SIMPLE TO CLASSIFY

→ COMPOSITE AND QUALITATIVE PROPERTIES

eg
- A "FUZZY" FABRIC (SURFACE TEXTURE, REFLECTIVITY, FIBER RIGIDITY)
- "SINTERABILITY" OF A CERAMIC POWDER (PARTICLE SIZE, SURFACE ENERGY, DIFFUSIVITY, VAPOR PRESSURE)

→ NON SINGLE-VALUED PROPERTIES (HYSTERITIC)
- POLARIZATION OF A FERROELECTRIC MATERIAL
- MAGNETIZATION OF A FERROMAGNETIC MATERIAL

→ PROPERTIES DETERMINED BY TESTS THAT ARE INHERENTLY NON-REVERSIBLE (THE PROPERTY MAY BE MEASURED FOR A SAMPLE THAT NO LONGER EXISTS!)
- YIELD STRENGTH
- FRACTURE TOUGHNESS
- STREAK COLOR

→ STRUCTURE-SENSITIVE PROPERTIES

PROPERTY DEPENDS ON DENSITY OR DISTRIBUTION OF DEFECTS

OUR PRESENT DISCUSSION WILL DEAL WITH <u>EQUILIBRIUM</u> <u>PROPERTIES</u> THAT ARE RIGOROUSLY DEFINED AND MEASUREABLE.

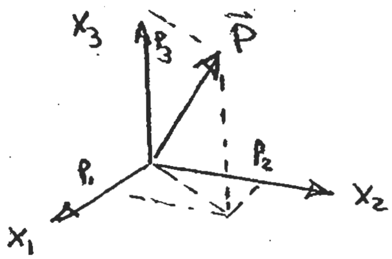
Our "Property" will express a response of the material to a generalized "force" by a generalized "displacement"

Examples: "Forces" : T (temperature) E ($\bar{E}$lectric field), σ (stress)

"Displacement" : S (entropy) D (displacement), ϵ (strain)

Tensors

Let us suppose that our generalized force is a vector, $\vec{P}$, whose components are P_1 P_2 and P_3 relative to a Cartesian (not crystallographic!!) coordinate system X_1 X_2 X_3



The response or "displacement" of the material is often a vector $\vec{q}$.

In many cases we can assume $p \propto q$ and write $p = a q$

This may be true, but is usually valid only for "small" q, and what constitutes "small" depends on the property.

Example ① The electrical conductivity σ of a ceramic dielectric material relates current density J (charge per unit area per unit time) to an applied electric field, $\mathcal{E}$ (volts/unit length) according to $J = \sigma \mathcal{E}$. But — make $\mathcal{E}$ sufficiently large and dielectric breakdown results

Example ② The magnetic susceptibility χ relates the magnetization M (magnetic moment per unit volume) to applied magnetic field H. For a dilute solution of Fe atoms in a silicate glass $M = \chi H$ But, if the magnetic field is sufficiently large, the magnetic moment of every Fe atom is dragged into alignment with H and one is unable to squeeze any more magnetization out of the material!

WE CAN RETAIN THE EXPECTATION THAT $|\vec{p}|$ WILL BE PROPORTIONAL TO $|\vec{q}|$ BUT RELAX THE ASSUMPTION THAT $\vec{p} \parallel \vec{q}$ BY ASSUMING THAT EACH COMPONENT OF $\vec{p}$ IS GIVEN BY A LINEAR COMBINATION OF EVERY COMPONENT OF $\vec{q}$. THAT IS, A RELATION OF THE FORM

$$\begin{cases} p_1 = a_{11} q_1 + a_{12} q_2 + a_{13} q_3 \\ p_2 = a_{21} q_1 + a_{22} q_2 + a_{23} q_3 \\ p_3 = a_{31} q_1 + a_{32} q_2 + a_{33} q_3 \end{cases}$$

WE CAN SUMMARIZE THE SET OF EQUATIONS WITH

$$p_i = \sum_{j=1}^{3} a_{ij} p_j$$

THE FIRST SUBSCRIPT, i, HAS MEANING: IT SPECIFIES WHICH COMPONENT p_i THAT ONE IS EVALUATING. THE SECOND SUBSCRIPT j IS AN INDEX OF SUMMATION, SOMETIMES REFERRED TO AS A "DUMMY INDEX".

WE WILL SHORTLY ENCOUNTER DOUBLE, TRIPLE OR QUADRUPLE (!) SUMMATIONS AND WILL VERY QUICKLY TIRE OF THE NEED TO CONSTANTLY WRITE Σ'S. LET US THEREFORE WRITE, SIMPLY

$$\boxed{p_i = a_{ij} p_j}$$

AND TAKE SUCH AN EXPRESSION TO MEAN THAT SUMMATION FROM 1 TO 3 OF ANY REPEATED INDEX (IN THIS EXAMPLE j) IS AUTOMATICALLY UNDERSTOOD.

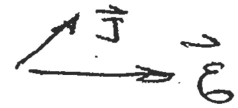
THIS IS REFERRED TO AS THE EINSTEIN CONVENTION

THE ARRAY OF COEFFICIENTS a_{ij} CONSTITUTES A TENSOR OF SECOND RANK (TWO SUBSCRIPTS) AND CAN RELATE TWO VECTORS (AS IN OUR EXAMPLES OF ELECTRONIC POLARIZABILITY AND THERMAL CONDUCTIVITY) OR CAN RELATE A SCALAR AND A SECOND RANK TENSOR (FOR EXAMPLE, THERMAL EXPANSION, α, RELATING STRAIN, ϵ_{ij} AND TEMPERATURE CHANGE, ΔT, ACCORDING TO $\epsilon_{ij} = \alpha_{ij} \Delta T$)

There is a second assumption built into the relation $p = aq$ because both p and q may be vectors:

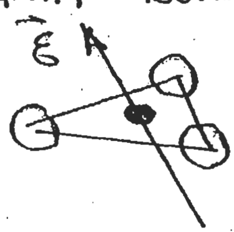
$$\vec{p} = a\vec{q}$$

We are thus assuming that $\vec{p}$ and $\vec{q}$ are parallel! Why not?!! Let's not be silly! Who would claim that if we apply an electric field $\vec{\mathcal{E}}$ to a material that the current runs off in a different direction!!



Well, we should perhaps not make this assumption too swiftly.

• Consider the electric dipole moment $\vec{p}$ that is induced on a tightly-bound triangular group of ions in a crystal in response to an applied electric field $\vec{\mathcal{E}}$. The property relating the two vectors is the electronic polarizability α



$\vec{p} = \alpha\vec{\mathcal{E}}$ But, would we expect the electron cloud on the ions to be displaced as easily within the plane of the group as in a direction normal to it?? Well, maybe not -----

• Consider the thermal conductivity of a single crystal of a layer structure, K. Thermal conductivity relates heat-flow density to an applied temperature gradient. K depends on the velocity of phonons in the structure and this can be several orders of magnitude larger in the plane of the layer than normal to them.

a_j, A VECTOR, IS A FIRST-RANK TENSOR AND SOME PROPERTIES ARE TENSORS OF FIRST RANK. AN EXAMPLE IS THE PYROELECTRICITY TENSOR THAT RELATES AN INDUCED POLARIZATION P_i (DIPOLE MOMENT PER UNIT VOLUME) TO A SCALER CHANGE IN TEMPERATURE, ΔT

AN ARRAY OF 3 X 9 OR 9 X 3 COEFFICIENTS a_{ijk} THAT EITHER RELATES A SECOND-RANK TENSOR TO A VECTOR, OR A VECTOR TO AN APPLIED SECOND-RANK TENSOR AS THE GENERALIZED FORCE CONSTITUTES A THIRD RANK TENSOR. EXAMPLES

- STRAIN ϵ_{ij} INDUCED BY AN APPLIED ELECTRIC FIELD $\mathcal{E}_k$ (THE CONVERSE PIEZOELECTRIC EFFECT)

$$\epsilon_{ij} = a_{ijk}\mathcal{E}_k$$

- POLARIZATION P_i (DIPOLE MOMENT PER UNIT VOLUME) INDUCED BY AN APPLIED STRESS σ_{jk}

$$P_i = a_{ijk}\sigma_{jk}$$

(THE DIRECT PIEZOELECTRIC EFFECT)

FOURTH-RANK TENSORS APPEAR IN ELASTICITY AS THE COEFFICIENTS THAT RELATE STRESS σ_{ij} AND STRAIN ϵ_{kl}

THE STIFFNESSES C_{ijkl} RELATE STRESS AND STRAIN

$$\sigma_{ij} = C_{ijkl}\,\epsilon_{kl}$$

(A 9X9 ARRAY OF 81 ELEMENTS!!)

THE COMPLIANCES S_{ijkl} RELATE STRAIN TO STRESS

$$\epsilon_{ij} = S_{ijkl}\,\sigma_{kl}$$

YET HIGHER-RANK TENSORS ARE REQUIRED TO DESCRIBE OTHER PROPERTIES — EG, THE CHANGE OF ELASTIC PROPERTIES WITH APPLIED PRESSURE!!

Transformation of Tensors

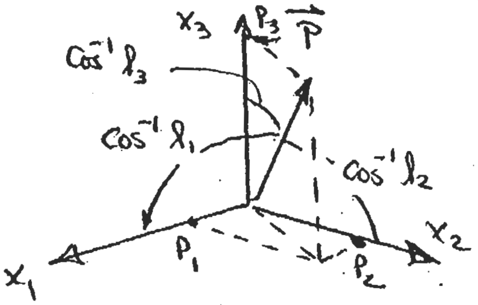
There are two implications of the presumably-satisfactory relation that we have proposed to relate two vectors

$$p_i = a_{ij} q_j$$

The first is that there is no reason to expect all nine elements a_{ij} to have the same numerical values.

- Therefore the components p_i and the magnitude of $\vec{p}$ will change as we change the orientation of the applied vector $\vec{q}$ (because the components of $\vec{q}$, $q_1 q_2 q_3$ depend in size on the orientation of $\vec{q}$.
  Accordingly, the property described by a_{ij} will be anisotropic! Its value will change with the orientation of the applied vector.
- The values of the applied vector $\vec{q}$ $(q_1\ q_2\ q_3)$ as well as the resulting vector $\vec{p}$ $(p_1\ p_2\ p_3)$ depend on the coordinate system $X_1 X_2 X_3$ that we use to decribe the vectors. The vectors are real, fixed in space, but if we change $X_1 X_2 X_3$ to some new axes $X_1'\ X_2'\ X_3'$ the numbers p_i and q_i change to new values. If p_i and q_i change then the tensor elements a_{ij} will also change with change in the coordinate system!

COMPONENTS of A VECTOR



LET'S CONSIDER A VECTOR of MAGNITUDE $|\vec{P}|$ THAT IS DIRECTED IN A PARTICULAR ORIENTATION THAT WE CAN SPECIFY BY MEANS of THE ANGLES Θ_1 Θ_2 & Θ_3 THAT IT MAKE WITH REFERENCE AXES X_1 X_2 X_3

THE THREE COMPONENTS of THE VECTOR ALONG THESE AXES WILL BE GIVEN BY

$$\begin{cases} P_1 = |P| \cos\Theta_1 \equiv |P| l_1 \\ P_2 = |P| \cos\Theta_2 \equiv |P| l_2 \\ P_3 = |P| \cos\Theta_3 \equiv |P| l_3 \end{cases}$$

THE QUANTITIES $\cos\Theta_i$ WILL OCCUR IN SO MUCH THAT WE ARE ABOUT TO DO THAT WE WILL INTRODUCE A SEPARATE SYMBOL l_i FOR THEM THAT WE WILL DIRECTLY USE TO SPECIFY AN ORIENTATION. WE WILL REFER TO THESE QUANTITIES AS DIRECTION COSINES

IN THE SPECIAL CASE IN WHICH $\vec{P}$ IS A VECTOR of UNIT MAGNITUDE $\vec{u}$, WHERE $|u| \equiv 1$, THE ABOVE SET of EQUATIONS REDUCES TO

$$\begin{cases} u_1 = |u| l_1 = l_1 \\ u_2 = |u| l_2 = l_2 \\ u_3 = |u| l_3 = l_3 \end{cases}$$

THIS MEANS THAT WE CAN REGARD THE DIRECTION COSINES AS THE COMPONENTS of A UNIT VECTOR THAT IS ORIENTED ALONG THE DIRECTION SPECIFIED BY l_i

THE MAGNITUDE of THIS UNIT VECTOR IS GIVEN BY

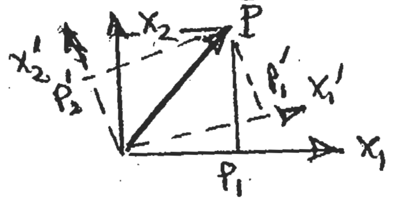
$$u_1^2 + u_2^2 + u_3^2 = |u|^2 = 1 = (|u| l_1)^2 + (|u| l_2)^2 + (|u| l_3)^2$$

$$\boxed{1 = l_1^2 + l_2^2 + l_3^2}$$

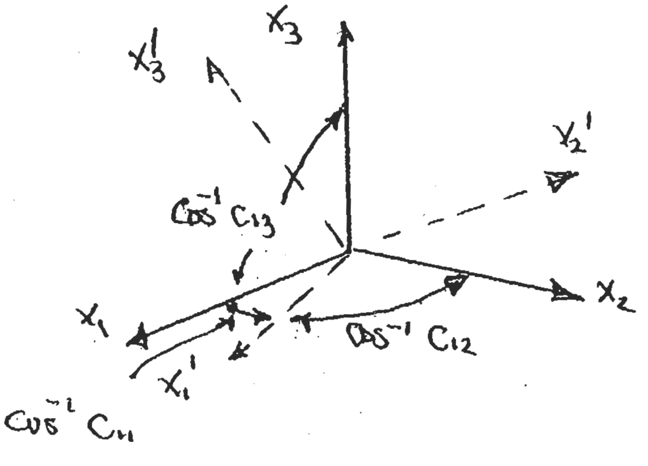
THE SUM of THE SQUARES of DIRECTION COSINES IS, THEREFORE, ALWAYS UNITY! ONLY TWO ARE INDEPENDENT. (NOTE, HOWEVER, THAT THE SIGN of THE THIRD DIRECTION COSINE MAY BE EITHER $\oplus$ OR $\ominus$)

SPECIFICATION OF A CHANGE IN COORDINATE SYSTEM

If we change from a first coordinate system $X_1 X_2 X_3$ to a new set of axes $X_1' X_2' X_3'$ the components of a vector $\vec{P}$, P_i, will change to new values P_i'. This is apparent from the two-dimensional example sketched to the left.



The vector $\vec{P}$ is real and remains majestically immobile in space. The components P_1 & P_2 or P_1' & P_2' however, depend on our choice of coordinate system. To specify a change in coordinate system it is necessary to provide the direction cosines for all three of the new axes $X_1' X_2' X_3'$ relative to the original set of axes $X_1 X_2 X_3$. This will require a 3×3 array of direction cosines that we will define as C_{ij} where C_{ij} is defined as the cosine of the angle between the new axis X_i' and the original reference axis X_j



Cos of angle between

AND	X_1	X_2	X_3
X_1'	C_{11}	C_{12}	C_{13}
X_2'	C_{21}	C_{22}	C_{23}
X_3'	C_{31}	C_{32}	C_{33}

RELATIONS AMONG THE COEFFICIENTS IN THE DIRECTION COSINE SCHEME FOR A CHANGE IN COORDINATE SYSTEM

WE HAVE SET UP A 3X3 MATRIX OF DIRECTION COSINES THAT SPECIFIES THE RELATION BETWEEN THE AXES OF NEW COORDINATE SYSTEM X_i' AND AN ORIGINAL SET OF AXES X_j:

$$\begin{cases} X_1' = C_{11}X_1 + C_{12}X_2 + C_{13}X_3 \\ X_2' = C_{21}X_1 + C_{22}X_2 + C_{23}X_3 \\ X_3' = C_{31}X_1 + C_{32}X_2 + C_{33}X_3 \end{cases}$$

OR $[X_i'] = [C_{ij}][X_j]$

IN MATRIX NOTATION

OR SIMPLY

$$X_i' = C_{ij}X_j$$

WHERE WE HAVE USED THE EINSTEIN CONVENTION IN WHICH SUMMATION FROM 1 TO 3 IS AUTOMATICALLY UNDERSTOOD IN ANY PRODUCT WITH REPEATED SUBSCRIPTS (IN THIS CASE, THE INDEX j)

A GREAT MANY RELATIONS EXIST AMONG THE DIRECTION COSINES C_{ij}. AS WE HAVE SEEN THAT DIRECTION COSINES CAN BE INTERPRETED AS THE THREE COMPONENTS OF A UNIT VECTOR POINTING IN A PARTICULAR DIRECTION. FROM WHICH:

(1) THE DIRECTION COSINES IN ANY ROW — FOR EXAMPLE

$$X_1' = C_{11}X_1 + C_{12}X_2 + C_{13}X_3$$

PROVIDES THE ORIENTATION OF THE NEW AXIS X_1' AND C_{11} IS THE COSINE OF THE ANGLE BETWEEN X_1' AND X_1, C_{12} THE COSINE OF THE ANGLE BETWEEN X_1' AND X_2 AND C_{13} THE COSINE OF THE ANGLE BETWEEN X_1' AND X_3'. IN OTHER WORDS C_{1j} REPRESENT THE COMPONENTS OF A UNIT VECTOR ALONG X_1' RELATIVE TO THE ORIGINAL AXES X_j. THEREFORE, THE SUM OF THEIR SQUARES $C_{11}^2 + C_{12}^2 + C_{13}^2$ MUST BE 1!

BY SIMILAR ANALYSIS WE CAN SAY THE SUM OF THE SQUARES OF THE ELEMENTS IN ANY ROW OF THE MATRIX C_{ij} IS UNITY!

(2) LET'S EXAMINE THE MEANING OF THE ELEMENTS OF ANY COLUMN IN $[C_{ij}]$ — FOR EXAMPLE, $\begin{matrix} C_{12} \\ C_{22} \\ C_{32} \end{matrix}$

C_{12} IS THE COSINES OF THE ANGLE BETWEEN X_1' AND X_2

C_{22} " " " " " X_2' AND X_2

C_{32} " " " " " X_3' AND X_2

THESE ARE THE DIRECTION COSINES OF X_2 IN THE ORIGINAL COORDINATE SYSTEM! THE SUM OF THEIR SQUARES MUST BE UNITY AND, IN GENERAL THE SUM OF THE SQUARES OF ELEMENTS IN ANY COLUMN IS UNITY

NOTE, HOWEVER, THAT THE RELATIONS IN ① AND ② HAVE LIMITATIONS: THEY INVOLVE THE SQUARES OF THE COSINES AND THEREFORE CANNOT BE USED TO DETERMINE THEIR SIGNS (WHICH, BEING COSINES, CAN BE ⊕ OR ⊖)

(3) IF THE ROW C_{1j} IN THE DIRECTION COSINE MATRIX REPRESENTS THE COMPONENTS THE DIRECTION COSINES OF X_1' RELATIVE TO $X_1 X_2 X_3$ AND, IF THE ROW C_{2j} REPRESENTS THE DIRECTION COSINES OF X_2' RELATIVE TO $X_1 X_2 X_3$ WE HAVE AT HAND THE COMPONENTS OF TWO UNIT VECTORS WHICH, IN CARTESIAN COORDINATE SYSTEMS, MUST BE ORTHOGONAL! THEREFORE, THE SCALAR (OR "DOT") PRODUCT OF THESE TWO VECTORS HAS TO BE ZERO!

THUS $C_{11} \cdot C_{21} + C_{12} \cdot C_{22} + C_{13} C_{23} = 0$

AND THE SUM OF THE PRODUCTS OF CORRESPONDING ELEMENTS IN ANY PAIR OF ROWS IN C_{ij} MUST BE ZERO

(4) BY SIMILAR ARGUMENT, COLUMNS C_{i1} AND C_{i2} REPRESENT THE COMPONENTS OF X_1 AND X_2 RELATIVE TO THE NEW COORDINATE SYSTEM $X_1' X_2' X_3'$ AND THE SUM OF THE PRODUCTS OF CORRESPONDING ELEMENTS IN ANY PAIR OF COLUMNS IN C_{ij} MUST BE ZERO

(5) THE TRANSFORMATION OF AXES $X_1 X_2 X_3 \rightarrow X_1' X_2' X_3'$ INVOLVES NO DISTORTION AND IS TERMED A "MEASURE PRESERVING" TRANSFORMATION. THE MATRIX $[C_{ij}]$ IS WHAT IS TERMED A UNITARY MATRIX AND HAS SEVERAL PROPERTIES, USEFUL IN PROBLEMS, THAT WE WILL STATE WITHOUT PROOF

- THE VALUE OF THE DETERMINANT $|C_{ij}| = \pm 1$

$+1$ IF THE NEW COORDINATE SYSTEM HAS NOT CHANGED CHIRALITY (THAT IS, RIGHT-HANDED → RIGHT HANDED OR LEFT HANDED → LEFT HANDED) AND -1 IF THE CHIRALITY IS CHANGED (RIGHT → LEFT HANDED OR LEFT → RIGHT HANDED

• Another very useful, labor-saving property of unitary matrices concerns the reverse transformation. That is, if

$$X_i' = C_{ij} X_j$$

what if we change our mind and want to go back from the new coordinate system to the original one? That relation is specified by the relation

$$X_i = [C_{ij}]^{-1} X_j'$$

where $[C_{ij}]^{-1}$ is the inverse matrix. For simple transformations of axes one can often write X_i in terms of X_j' by inspection and pick off the elements of C_{ij}^{-1} directly.

One can also evaluate $[C_{ij}]^{-1}$ more generally, but ----

This involves an unpleasant amount of evaluation of 3x3 determinants for a total of nine elements!

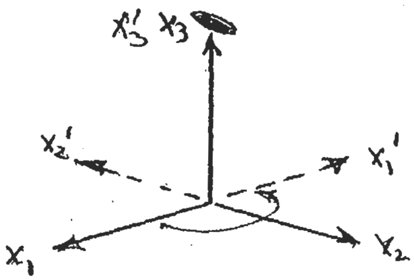
The great assistance of a unitary matrix in this situation is that

$$[C_{ij}]^{-1} = [C_{ji}]$$

Note that C_{ji} is simply the 3x3 array of coefficients flipped across the diagonal that runs from upper left to lower right. This is called the transpose of the matrix C_{ij} and is written either $[\tilde{C}_{ij}]$ or $[C_{ij}]^T$ by various writers. (We shall use the former notation.)

Obviously specifying $[\tilde{C}_{ij}]$ is something that we can do in a twinkling of the eye by comparison with the evaluation of nine ratios of 3x3 determinants!!

Moment of Reassurance



All this may seem rather daunting, but we shall shortly use this formalism to obtain constraints imposed on property tensors by crystal symmetry. For a 2-fold axis, for example,

$$\begin{cases} X_1' = -X_1 \\ X_2' = -X_2 \\ X_3' = X_3 \end{cases}$$

so for this transformation $[C_{ij}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

STILL MORE TRANSFORMATION MACHINERY

TRANSFORMATION OF A SECOND RANK TENSOR

SUPPOSE A GENERALIZED VECTOR DISPLACEMENT, q_i, IS RELATED TO A GENERALIZED VECTOR FORCE, P_j, BY A SECOND RANK TENSOR a_{ij}:

$$q_i = a_{ij} P_j$$

IF THE COORDINATE SYSTEM X_i RELATIVE TO WHICH THE COMPONENTS OF $\vec{q}$ AND $\vec{p}$ ARE DEFINED IS CHANGED TO SOME NEW SET OF BASIS VECTORS X_i' SPECIFIED BY THE DIRECTION COSINE SCHEME $[C_{ij}]$, THEN THE COMPONENTS OF $\vec{q}$ AND $\vec{p}$ TAKE ON NEW NUMERICAL VALUES q_i' AND P_j'. OUR TENSOR RELATION, HOWEVER, SAYS THAT EACH COMPONENT OF $\vec{q}$ IS GIVEN BY A LINEAR COMBINATION OF ALL THREE COMPONENTS OF $\vec{p}$. IF THE COMPONENTS OF $\vec{q}$ AND $\vec{p}$ HAVE CHANGED THEIR NUMERICAL VALUES, THE ELEMENTS OF THE TENSOR WHICH RELATES THEM MUST ALSO CHANGE TO SOME NEW VALUES SUCH THAT

$$q_i' = a_{ij}' P_j'$$

HOW MAY a_{ij}' BE EVALUATED IN TERMS OF THE ORIGINAL ELEMENTS a_{ij} AND THE DIRECTION COSINE SCHEME $[C_{ij}]$ WHICH SPECIFIES THE CHANGE OF REFERENCE AXES?

LETS BEGIN BY WRITING THE NEW COMPONENTS q_i' IN TERMS OF THE ORIGINAL COMPONENTS q_j

$$q_i' = C_{im} q_m$$

q_m IS DETERMINED BY THE TENSOR RELATION WHICH, FOR THE M^{th} COMPONENT OF $\vec{q}$, SPECIFIES $q_m = a_{m\ell} P_\ell$. THUS

$$q_i' = C_{im} [a_{m\ell} P_\ell]$$

NOW, IN ORDER TO OBTAIN AN EXPRESSION WHICH HAS q' ON THE LEFT AND P' ON THE RIGHT, LET'S WRITE P_ℓ IN TERMS OF THE NEW COMPONENTS OF $\vec{p}$. IF $[P_\ell'] = [C_{\ell j}][P_j]$ THEN $[P_\ell] = [C_{\ell j}]^{-1}[P_j'] = [\tilde{C}_{\ell j}][P_j'] = [C_{j\ell}][P_j']$ AS THE INVERSE OF A UNITARY MATRIX IS EQUAL TO THE TRANSPOSE OF THE MATRIX. SUBSTITUTING $C_{j\ell} P_j'$ FOR P_ℓ WE OBTAIN

$$q_i' = C_{im} [a_{m\ell} (C_{j\ell} P_j')]$$

THE ABOVE EXPRESSION REPRESENTS A <u>TRIPLE</u> SUMMATION (OVER j, ℓ & m, AS THESE SUBSCRIPTS ARE REPEATED) OF A PRODUCT OF FOUR QUANTITIES. WE MAY PERFORM THE SUMMATION IN ANY ORDER WE WISH AND, SIMILARLY, ARRANGE THE PRODUCT IN ANY

DESIRED ORDER. THEREFORE, LETS WRITE BY REARRANGEMENT

$$q_i' = [C_{im} C_{jl} a_{ml}] p_j'$$

BUT, BY DEFINITION,

$$q_i' = a_{ij}' p_j'$$

THUS

$$\boxed{a_{ij}' = C_{im} C_{jl} a_{ml}}$$

THIS EXPRESSION IS A DOUBLE SUMMATION OVER m & l; THUS EACH ELEMENT IN THE NEW TENSOR $[a_{ij}']$ IS GIVEN BY A LINEAR COMBINATION OF EVERY ONE OF THE NINE ELEMENTS OF THE ORIGINAL TENSOR $[a_{ml}]$

NOTE THAT m & l HAVE NO PHYSICAL SIGNIFICANCE AND FUNCTION MERELY AS INDICES OF SUMMATION; THEY ARE "DUMMY INDICES". THE INDICES i & j, IN CONTRAST, (WHICH APPEAR, RESPECTIVELY, AS THE FIRST INDEX IN THE TWO ELEMENTS OF THE DIRECTION COSINE MATRIX IN THE PRODUCT) DO HAVE ABSOLUTE SIGNIFICANCE: THEY ARE SPECIFIED BY THE SUBSCRIPTS OF THE PARTICULAR TRANSFORMED TENSOR ELEMENT a_{ij}' WHICH ONE WISHES TO EVALUATE.

TRANSFORMATION OF HIGHER-RANK TENSORS

THE ABOVE RESULT MAY BE EXTENDED, USING SIMILAR REASONING, TO THE LAW FOR TRANSFORMATION OF A TENSOR OF ANY RANK. THE RESULT IS

$$\boxed{a_{ijkl\ldots}' = \underbrace{C_{iI} C_{jJ} C_{kK} C_{lL} \cdots}_{\text{THE NUMBER OF DIRECTION COSINES IN THE COEFFICIENT IS EQUAL TO THE RANK OF THE TENSOR}} \underbrace{a_{IJKL\ldots}}_{\text{IJKL — ARE "DUMMY" INDICES OF SUMMATION.}}}$$

$ijkl$ — ARE "REAL": DETERMINED BY THE ELEMENT YOU WISH TO EVALUATE.

NOTE THAT, INDEPENDENT OF THE RANK OF THE TENSOR, EACH NEW, TRANSFORMED ELEMENT IS GIVEN BY A LINEAR COMBINATION OF EVERY ELEMENT OF THE ORIGINAL TENSOR.

THE COEFFICIENT IN FRONT OF EACH ORIGINAL ELEMENT IS A PRODUCT OF DIRECTION COSINES, EQUAL IN NUMBER TO THE RANK OF THE TENSOR.

THUS, GENERAL TRANSFORMATION OF, SAY, THE ELASTIC STIFFNESSES (A 4th RANK TENSOR) REQUIRES A SUMMATION OF A PRODUCT OF FIVE TERMS FOR ALL $9 \times 9 = 81$ ELEMENTS OF THE TENSOR, AND REPETITION OF THE PROCESS 81 TIMES FOR THE NEW ELEMENTS! $(5 \times 81) \cdot 81 =$ 32,805 TERMS!