Section 1 SOLUTIONS

SOLUTIONS - SECTION 1

1A-1 a) $y = c_1 e^x + c_2 x e^x$

$\times -2$ $\quad y' = (c_1 + c_2)e^x + c_2 x e^x$

$y'' = (c_1 + 2c_2)e^x + c_2 x e^x$

Add $\quad y'' - 2y' + y = 0$ ✓ (easily checked)

b) $y' = -\dfrac{(\sin x + a)}{x^2} + \dfrac{\cos x}{x} + \sin x$

$\dfrac{y}{x} = \dfrac{\sin x + a}{x^2} - \dfrac{\cos x}{x}$

$\therefore\ y' + \dfrac{y}{x} = \sin x$

1A-2 a) $c_1 e^{kx}$ and $c_1' e^{k'x}$ are the same only if $c_1 = c_1'$, $k = k'$

b) let $k = c_1 e^a$

then $y = k e^x$

c) $\cos 2x = \cos^2 x - \sin^2 x$
$= 2\cos^2 x - 1$

$\therefore\ y = c_1 + c_2(2\cos^2 x - 1) + c_3 \cos^2 x$
$= (c_1 - c_2) + (2c_2 + c_3)\cos^2 x$
$= k_1 + k_2 \cos^2 x$

d) $y = \ln(ax+b)(cx+d)$
$= \ln(acx^2 + (ad+bc)x + bd)$

$\therefore\ y = \ln(k_1 x^2 + k_2 x + k_3)$

1A-3 a) Separating variables gives

$y^2\,dy = \dfrac{dx}{\ln x}$ $\quad$ Integrate both sides from 2 to x:

$\left.\dfrac{y^3}{3}\right]_2^x = \displaystyle\int_2^x \dfrac{dt}{\ln t}$ . . Now use $y(2)=0$:

$\dfrac{y(x)^3}{3} - \dfrac{0^3}{3} = \displaystyle\int_2^x \dfrac{dt}{\ln t}$

$\therefore\ y = \left[3\displaystyle\int_2^x \dfrac{dt}{\ln t}\right]^{1/3}$.

b) Separate variables: $\dfrac{dy}{y} = \dfrac{e^x}{x}dx$

Can either use same method as in (a), or else: integrate both sides, using a definite integral as the antiderivative on the right:

$\ln y + c = \displaystyle\int_1^x \dfrac{e^t}{t}dt$ ⊛

Evaluate c by using $y(1) = 1$. This gives

$\underbrace{\ln y(1)}_{0} + c = \displaystyle\int_1^1 \dfrac{e^t}{t}dt = 0$

$\therefore\ c = 0$

So from ⊛ $\quad y = e^{\int_1^x e^t/t\,dt}$

1A-4 a) $\dfrac{y\,dy}{y+1} = x\,dx$ $\quad$ Integrate, noting that $\dfrac{y}{y+1} = 1 - \dfrac{1}{y+1}$

$\therefore\ dy - \dfrac{dy}{y+1} = x\,dx$

$y - \ln(y+1) = c + \frac{1}{2}x^2$ $\quad$ Put $x=2$ to evaluate c: $[y(2)=0]$

$0 - \ln(1) = c + \frac{1}{2}\cdot 2^2$

$\therefore\ c = -2$

soln: $\boxed{y - \ln(y+1) = \frac{1}{2}x^2 - 2}$

b) $\sec^2 u\,du = \sin t\,dt$

$\therefore\ \tan u = -\cos t + c$ $\quad$ Put $t=0$: $u(0)=0$

$\therefore\ \tan 0 = -1 + c$

so $c = 1$

soln: $\boxed{u = \tan^{-1}(1 - \cos t)}$

$\boxed{1A-5}$ a) $\dfrac{dy}{y^2-2y} = -\dfrac{dx}{x^2}$ Integrate left side by partial fractions

$$\frac{1}{2}\frac{dy}{y-2} - \frac{1}{2}\frac{dy}{y} = \frac{-dx}{x^2}$$

$$\frac{1}{2}\ln\left(\frac{y-2}{y}\right) = c_1 + \frac{1}{x}$$ Multiply by 2, exponentiate

$$= 1 - \frac{2}{y} \rightarrow \frac{y-2}{y} = c_2 e^{2/x}$$

$$\therefore \quad y = \frac{2}{1 - c_2 e^{2/x}}$$ algebra; (replace left side by $1-\frac{2}{y}$)

b) $$\frac{dv}{\sqrt{1-v^2}} = \frac{dx}{x}$$

$$\sin^{-1} v = \ln x + c$$

$$v = \sin(\ln x + c)$$

c) $$\frac{dy}{(y-1)^2} = \frac{dx}{(x+1)^2}$$

$$-\frac{1}{y-1} = \frac{-1}{x+1} + c$$

Solve for y by ordinary algebra.

$$y = 1 + \frac{x+1}{1 - c(x+1)}$$

d) $$\frac{dx}{\sqrt{1+x}} = \frac{dt}{t^2+4}$$

$$2\sqrt{1+x} = \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + c$$

$$\therefore \quad x = \frac{1}{4}\left(\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + c\right)^2 - 1$$

These problems all take for granted that you know the standard integration formulas and methods from 18.01. Review them if you are having trouble.

You need also the laws of exponentials and logarithms.

$\boxed{1B-1}$

a) $\dfrac{\partial M}{\partial y} = 3x^2 = \dfrac{\partial N}{\partial x}$ $\therefore$ exact. what's $f(x,y)$?

$$\frac{\partial f}{\partial x} = 3x^2 y \quad \therefore f = x^3 y + g(y)$$

$$\frac{\partial f}{\partial y} = x^3 + g'(y) = x^3 + y^3. \quad \therefore g = \tfrac{1}{4}y^4 + c$$

so that $f = x^3 y + \frac{1}{4}y^4 + c$.

Solution: $\boxed{x^3 y + y^4/4 = c_1}$

b) $\dfrac{\partial M}{\partial y} = -2y$, $\dfrac{\partial N}{\partial x} = -2x$ not exact.

c) $\dfrac{\partial M}{\partial v} = e^{uv} + ve^{uv} = \dfrac{\partial N}{\partial u}$ $\therefore$ exact

$$\frac{\partial f}{\partial u} = ve^{uv}, \quad \therefore f = e^{uv} + g(v)$$

$$\frac{\partial f}{\partial v} = ue^{uv} + g'(v) = ue^{uv}. \quad \therefore g = c$$

so $f = e^{uv} + c$. Soln: $e^{uv} = c_1$

or taking ln of both sides:

$\boxed{uv = c_2}$

d) $\dfrac{\partial M}{\partial y} = 2x$, $\dfrac{\partial N}{\partial x} = -2x$ not exact.

$\boxed{1B-2}$

a) Multiply by y — this gives

$$2xy\,dx + x^2 dy = 0$$

or $d(x^2 y) = 0$ $\therefore$ $x^2 y = c$

so $\boxed{y = c/x^2}$

b) Integrating factor is $\dfrac{1}{y^2}$:

$$\frac{y\,dx - x\,dy}{y^2} - \frac{dy}{y} = 0$$

$$d\left(\frac{x}{y}\right) - d(\ln y) = 0$$

$$(*) \quad \frac{x}{y} - \ln y = c.$$

Evaluate c by setting $x = 1$ (so $y(1) = 1$)

$$\therefore \frac{1}{1} - \underbrace{\ln 1}_{0} = c, \quad \text{so} \quad c = 1$$

$$\therefore x - y\ln y = y$$

or $\boxed{x = y(\ln y + 1)}$

1B-2

c) Divide by t^2 (so integrating factor is $1/t^2$)

$$\left(1+\frac{4}{t^2}\right)dt = \frac{x\,dt - t\,dx}{t^2}$$

$$\therefore\ d\left(t-\frac{4}{t}\right) = d\left(\frac{-x}{t}\right)$$

$$t - 4/t = \frac{-x}{t} + c$$

$$\therefore\ \boxed{x = 4 - t^2 + ct}$$

d) $\frac{1}{u^2+v^2}$ is an integrating factor:

$$\frac{u\,du + v\,dv}{u^2+v^2} + \frac{v\,du - u\,dv}{u^2+v^2} = 0$$

$$\tfrac{1}{2}\ln(u^2+v^2) + \tan^{-1}\left(\frac{u}{v}\right) = c$$

when $u=0$, $v=1$; $\frac{1}{2}\ln 1 + \tan^{-1}(0) = c$ $\therefore\ c=0$

$$\boxed{\tfrac{1}{2}\ln(u^2+v^2) + \tan^{-1}\left(\frac{u}{v}\right) = 0}$$

(substitute $r = \sqrt{u^2+v^2}$, $\theta = \tan^{-1}\frac{u}{v}$ to get polar coords)

equation becomes $\ln r + \theta = 0$

or $\boxed{r = e^{-\theta}}$

1B-3

a) $z = y/x$ $\therefore\ y = zx$, $y' = z'x + z$

Substituting:

$$z'x + z = \frac{2z-1}{z+4}, \quad \therefore\ z'x = -\frac{(z+1)^2}{z+4}$$

Sep. variables:

$$\frac{z+4}{(z+1)^2}\,dz = \frac{-dx}{x}$$

For ease, write $z+1 = u$

$$\left(\frac{u+3}{u^2}\right)du = \frac{-dx}{x}$$

integrate:

$$\ln u - \frac{3}{u} = -\ln x + c$$

To improve this:

$$\ln u + \ln x = \frac{3}{u} + c$$

Combine + exponentiate: $ux = ke^{3/u}$

Finally: $u = z+1 = \frac{y}{x} + 1 = \frac{y+x}{x}$

$$\therefore\ \boxed{y + x = ke^{3x/y+x}}$$

b) let $z = \frac{w}{u}$, so $w = zu$, $w' = z'u + z$

Substituting:

$$z'u + z = \frac{2z}{1-z^2}$$

$$\therefore\ z'u = \frac{z(1+z^2)}{1-z^2}, \text{ after a little algebra}$$

Separate variables:

$$(*)\quad \frac{1-z^2}{z(1+z^2)}\,dz = \frac{du}{u}.$$

Use partial fractions on the left; result

$$\frac{1-z^2}{z(1+z^2)} = \frac{1}{z} + \frac{-2z}{z^2+1}$$

Integrating $(*)$:

$$\ln z - \ln(z^2+1) = \ln u + c$$

Combine and exponentiate both sides:

$$\frac{z}{z^2+1} = ku$$

Finally, put $z = w/u$; result is

$$\boxed{\frac{w}{w^2+u^2} = k}$$

as the solution (you could also solve for u in terms of w)

c) Put $z = y/x$; so $y = zx$, $y' = z'x + z$

Here $\frac{dy}{dx} = \frac{y^2 + x\sqrt{x^2-y^2}}{xy}$ Substitute $y = zx$

$$z'x + z = \frac{z^2 + \sqrt{1-z^2}}{z}.$$

$$\therefore\ z'x = \frac{\sqrt{1-z^2}}{z}$$

Separate variables

$$\frac{z\,dz}{\sqrt{1-z^2}} = \frac{dx}{x}$$

$$-\sqrt{1-z^2} = \ln x + c$$

$$\boxed{\sqrt{1 - y^2/x^2} = c_1 - \ln x}$$

This can be solved explicitly for y: square both sides, etc...

$$\boxed{y = x\sqrt{1-(c_1 - \ln x)^2}}$$

1B-4

$y = ux^n$

$\therefore\ y' = x^n u' + n x^{n-1} u$

$$x^n u' + n x^{n-1} u = \frac{4 + x^{2n+1} u^2}{x^{n+2} u}$$

$$\therefore\ u' = \frac{4 + (1-n) x^{2n+1} u^2}{x^{2n+2} u}$$

If $n=1$, we can separate vars:

$$u\,du = \frac{4\,dx}{x^4}$$

$$\therefore\ \frac{u^2}{2} = \frac{-4}{3}\cdot\frac{1}{x^3} + c$$

Since $n=1$, $u = y/x$

$$\therefore\ \boxed{y^2 = -\frac{8}{3x} + 2cx^2}$$

1B-5

a) $y' + \frac{2}{x} y = 1$ when written in normal form for linear eq'n.

Integ. factor: $e^{\int 2/x\,dx} = e^{2\ln x} = x^2$

$\therefore\ x^2 y' + 2xy = x^2$

or $(x^2 y)' = x^2$

$x^2 y = \frac{1}{3}x^3 + c$

$$\boxed{y = \frac{x}{3} + \frac{c}{x^2}}.$$

b) In standard form;

integ. factor is $e^{\int -\tan t\,dt} = e^{\ln(\cos t)}$

$= \cos t$

$\therefore\ \cos t \frac{dx}{dt} - x \sin t = t$

or $(x\cos t)' = t$

$x \cos t = \frac{t^2}{2} + c$

Since $x(0) = 0$, putting $t=0$ shows $c=0$.

$$\therefore\ \boxed{x = \frac{t^2}{2}\sec t}$$

1B-5

c) $(x^2-1)y' + 2xy = 1$ LHS is already exact!

$[(x^2-1)y]' = 1$

$(x^2-1)y = x + c$

$$\therefore\ y = \frac{x+c}{x^2-1}$$

d) Writing it in standard linear form

$$\frac{dv}{dt} + \frac{3v}{t} = 1$$

Integrating factor: $e^{\int 3/t\,dt} = e^{3\ln t} = t^3$

$\therefore\ t^3 v' + 3t^2 v = t^3$

$(t^3 v)' = t^3$

$t^3 v = \frac{1}{4}t^4 + c$

$v(1) = \frac{1}{4} \Rightarrow c = 0$ (put $t=1$)

$$\therefore\ \boxed{v = \frac{1}{4}t}$$

1B-6

The integrating factor for this linear equation is $e^{\int a\,dt} = e^{at}$

$(xe^{at})' = e^{at} r(t)$

$$x = e^{-at}\left[\int_0^t e^{as} r(s)\,ds\right] + c$$

$$x = \frac{\int_0^t e^{as} r(s)\,ds}{e^{at}} + \frac{c}{e^{at}}$$

To find $\lim_{t\to\infty} x(t)$, use L'Hospital's rule, (∞/∞) [note that $c/e^{at} \to 0$] differentiating top and bottom

$$\therefore\ \lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{e^{at} r(t)}{a e^{at}} = \lim_{t\to\infty} \frac{r(t)}{a}$$

$= 0$ by hypothesis

[Where did we need the hypothesis $a>0$?]

[We used, in connection with L'H rule, the result $\frac{d}{dt}\int_0^t e^{as} r(s)\,ds = e^{at} r(t)$.

This follows from the 2^{nd} Fundamental Theorem of calculus].

1B-7

$$\frac{dy}{dx} = \frac{y}{y^3+x} \Rightarrow \frac{dx}{dy} = \frac{y^3+x}{y}$$

$$\therefore \frac{dx}{dy} - \frac{1}{y}x = y^2$$

This is now a linear equation in x.

Integ. factor: $e^{-\int \frac{dy}{y}} = e^{-\ln y} = y^{-1}$

$\therefore$ multiply by $\frac{1}{y}$:

$$\frac{1}{y}\frac{dx}{dy} - \frac{1}{y^2}x = y$$

or

$$\frac{d}{dy}\left(\frac{x}{y}\right) = y$$

$$\frac{x}{y} = \frac{y^2}{2} + c$$

$$\boxed{x = y^3/2 + cy}$$

1B-8

The systematic procedure — it always works, though it's a bit longer in this case -: since we want to substitute for y, y', begin by expressing them in terms of u.

(Don't just differentiate $u = y^{1-n}$ as is).

$$y = u^{1/1-n}$$

$$y' = \frac{1}{1-n}u^{\left(\frac{1}{1-n}-1\right)} \cdot u' = \frac{1}{1-n}u^{\frac{n}{1-n}}u'$$

Substitute into the ODE:

$$\frac{1}{1-n}u^{\frac{n}{1-n}}u' + pu^{\frac{1}{1-n}} = qu^{\frac{n}{1-n}}$$

Divide through by $u^{\frac{n}{1-n}}$:

$$\boxed{\frac{1}{1-n}u' + pu = q}$$

[Note: in this particular case, it's actually easier just to bumble around, but in general, this only leads to a mess.

However: $y' + py = qy^n$

Divide: $\frac{y'}{y^n} + \frac{p}{y^{n-1}} = q$ (*)

Put $u = y^{1-n} = \frac{1}{y^{n-1}}$

$u' = (1-n)\cdot\frac{1}{y^n}\cdot y'$

$\therefore$ (*) becomes

$\frac{u'}{1-n} + pu = q$, as before.]

1B-9

$n=2$, so $u = y^{1-2} = y^{-1}$ (by problem 13)

Since we want to substitute for y, y', express them in terms of u and u':

$$y = \frac{1}{u}, \quad y' = -\frac{1}{u^2}\cdot u'$$

$\therefore$ the ODE becomes

$$-\frac{u'}{u^2} + \frac{1}{u} = 2x \cdot \frac{1}{u^2}$$

or $\boxed{u' - u = -2x}$ in standard linear eqn form.

Integ. factor: $e^{\int -dx} = e^{-x}$

Eq'n becomes

$$(e^{-x}u)' = -2xe^{-x} \leftarrow \text{integrate by parts}$$

$$\therefore e^{-x}u = 2xe^{-x} + 2e^{-x} + c$$

$$u = 2x + 2 + Ce^x$$

$$\therefore \boxed{y = \frac{1}{2x+2+Ce^x}}$$

1B-9

$y' - y$ Here $n=3$, so by prob. 13,

$$u = y^{1-3} = y^{-2}$$

As above, calculate y, y' in terms of u and u' (not other way around)

$$y = \frac{1}{\sqrt{u}}, \quad y' = -\frac{1}{2}u^{-3/2}\cdot u'$$

Substitute into the ODE:

$$-x^2\cdot\frac{u'}{2u^{3/2}} - \frac{1}{u^{3/2}} = \frac{x}{u^{1/2}}$$

$$\therefore \boxed{u' + \frac{2u}{x} = -\frac{2}{x^2}}$$

This is linear ODE; integ. factor is

$$e^{\int\frac{2dx}{x}} = e^{2\ln x} = x^2$$

ODE becomes

$$x^2u' + 2xu = -2$$

$$(x^2u)' = -2$$

$$x^2u = -2x + c$$

$$u = \frac{c-2x}{x^2}$$

$$\boxed{y = \frac{\pm x}{\sqrt{c-2x}}}$$

1B-10

a) $y = y_1 + u$

$y' = y_1' + u' = A + By_1 + Cy_1^2 + u'$

Substituting into the ODE:

$$A + By_1 + Cy_1^2 + u' = A + B(y_1+u) + C(y_1+u)^2$$

After some algebra,

$$u' = Bu + 2Cy_1u + Cu^2$$

$$\therefore \; u' - (B + 2Cy_1)u = Cu^2$$

This is a Bernouilli eq'n (problem 13) with $n = 2$.

b) By inspection, $y_1 = x$ is a sol'n to the ODE. $\therefore$ put $y = x + u$

$y' = 1 + u'$;

Substitution into the ODE gives

$$1 + u' = 1 - x^2 + (x+u)^2$$

$$\therefore \; \boxed{u' - 2xu = u^2},$$

a Bernouilli equation with $n = 2$.

Put $w = u^{1-2} = u^{-1}$

$\therefore \; u = \frac{1}{w}, \quad u' = \frac{-w'}{w^2}$

Substituting,

$$\frac{-w'}{w^2} - \frac{2x}{w} = \frac{1}{w^2}$$

or $\boxed{w' + 2xw = -1}$

Linear ODE with integrating factor $e^{\int 2x\,dx} = e^{x^2}$

$$\therefore \; (e^{x^2}w)' = -e^{x^2}$$

$$e^{x^2}w = -\int e^{x^2}dx + c$$

$$\boxed{w = -e^{-x^2}\int e^{x^2}dx + Ce^{-x^2}}$$

Finally:

$$y = x + u = x + \frac{1}{w}$$

$$\therefore \; y = x + \frac{e^{x^2}}{c - \int e^{x^2}dx}$$

(Actually, no value for C gives the original sol'n $y = x$; we have to take "$C = \infty$", or simply add $y = x$ to the above family.

1B-11

a) $y' = z$

$$y'' = \frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dx} = \frac{dz}{dy}\cdot z$$

Substitute into the ODE:

$$\frac{dz}{dy}\cdot z = a^2y \; ; \quad \text{Sep. vars:}$$

$$z\,dz = a^2y\,dy$$

$$z^2 = a^2y^2 + K$$

$$z = \sqrt{a^2y^2 + K}$$

$$\therefore \; y' = \sqrt{a^2y^2+K}$$

Separate variables again:

$$\frac{dy}{\sqrt{y^2 + K/a^2}} = a\,dx$$

Look this integral up!

$$\cosh^{-1}\left(\frac{ay}{\sqrt{K}}\right) = ax + c$$

$$y = \frac{\sqrt{K}}{a}\cosh(ax + c)$$

$$\therefore \; \boxed{y = c_1\cosh(ax + c)}$$

1B-11

(b) Let $y' = z$

$$y'' = \frac{dz}{dx} = \frac{dz}{dy}\cdot z$$

Substituting, $y\cdot\frac{dz}{dy}\cdot z = z^2$

$$\therefore \frac{dz}{z} = \frac{dy}{y} \qquad \therefore \ln z = \ln y + \text{const.}$$

$$\therefore \quad z = y' = Ky$$

Then $\frac{dy}{y} = K\,dx$

$$\therefore \ln y = Kx + c$$

or $y = e^{Kx+c}$ is the solution

1B-11

(c) Let $y' = z$

$$y'' = \frac{dz}{dy}\cdot z$$

Substituting, $\frac{dz}{dy}\cdot z = z(1+3y^2)$

$$\therefore dz = (1+3y^2)dy$$

$$\therefore z = y + y^3 + c$$

Using the initial conditions, $c = 0$

$$\therefore \frac{dy}{y+y^3} = dx \qquad \left(\text{remember: } z = \frac{dy}{dx}\right)$$

Integrating by partial fractions:

$$\frac{1}{y+y^3} = \frac{1}{y(y^2+1)} = \frac{1}{y} - \frac{y}{y^2+1}$$

$$\therefore \frac{dy}{y} - \frac{y\,dy}{y^2+1} = dx$$

$$\ln y - \tfrac{1}{2}\ln(y^2+1) = x + c$$

Exponentiating both sides,

$$\frac{y}{\sqrt{y^2+1}} = Ke^x$$

Using the initial conditions,

$$\frac{1}{\sqrt{2}} = K$$

$\therefore$ soln: $\boxed{\frac{y}{\sqrt{y^2+1}} = \frac{e^x}{\sqrt{2}}}$ (can solve for y in terms of x, if desired)

$$y = \frac{e^x}{\sqrt{2-e^{2x}}}$$

(by squaring both sides)

$\boxed{1B-12}$

1. Exact; also linear (divide by dx)

2. Linear; (integ. factor is e^{t^2})

3. Homogeneous: put $z = y/x$, get an ODE for z where you separate variables.

4. Separate variables; also linear in q and linear in p.

5. Exact; also linear.

6. Separate variables.

7. Bernoulli equation: $n=-1$
   put $u = y^{1-(-1)} = y^2 \ldots$

8. Separate variables: $\dfrac{dv}{e^{3v}} = e^{2u}du$

9. Divide by x – this make it homogeneous, so put $z = y/x \ldots$

10. Linear equation (integ. factor is $\frac{1}{x^2}$)

11. Think of y as indep't variable, x as dep't variable; then equation is $\dfrac{dx}{dy} = x + e^y$, which is linear in x.

12. Separate variables; also a Bernoulli equation (exercise 13)

13. When written in the form $P(x,y)dx + Q(x,y)dy = 0$, it becomes exact.

14. Linear, with int. factor e^{3x}

15. Divide by x — it becomes homogeneous, so put $z = y/x$, etc.

16. Separate variables

17. Riccati equation (exercise 15a)
A particular sol'n is $y_1 = x^2$; make the substitution $u = y - y_1$, get Bernoulli eq'n in u $(n=2)$, etc.

18. Autonomous – x missing.
Put $y' = v$, $y'' = v\dfrac{dv}{dy}$; separate variables

19. homogeneous – put $z = s/t$
($\ln s - \ln t = \ln s/t$, notice)

20. Exact when written as $Pdy + Qdx = 0$

21. Bernoulli eqn with $n=2$. (ex. 13)

22. Make change of variable
$$u = x + y$$
$$(\text{so } u' = 1 + y')$$
Then you can separate variables

23. Becomes linear if you think of y as indep't variable, s as dependent variable.

24. Linear (u dep't variable, t indep't variable)

25. $y_1 = -x$ is a particular sol'n.
Riccati equation (ex. 15a) –
put $u = y - y_1, \ldots$

OR BETTER:
write as $y' + (x+y)^2 + (x+y) + 1 = 0$.
and put $u = x + y$, $u' = 1 + y'$
leads to separation of variables.

26. Put $y' = v$ (so $y'' = v'$)
Get a first order linear eq'n in v.

1C-1

(a)

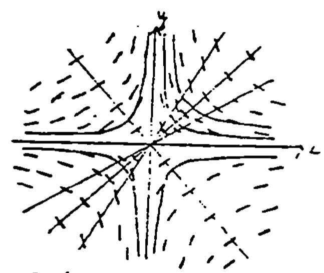
Isoclines: $-\frac{y}{x} = c$

Exact solution:

$$\frac{dy}{y} = \frac{-dx}{x}$$

$$\therefore \ln y = -\ln x + K'$$

$$\therefore y = \frac{K'}{x}$$



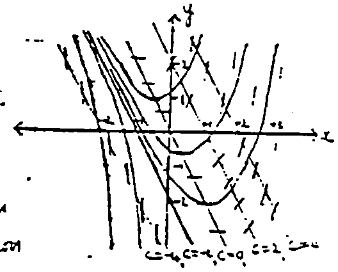
(Sol'n curves are hyperbolas)

(b) Isoclines:

$2x + y = c$.

This is a solution if $y' = -2 = c$;

i.e. $y + 2x + 2 = 0$ is an isocline which is a solution

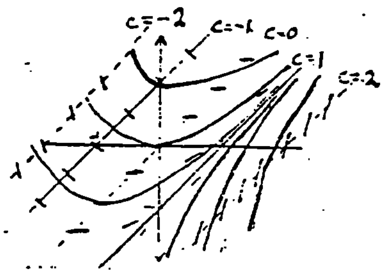


(c) Isoclines:

$x - y = c$

This is a solution if $y' = 1 = c$;

i.e., $x - y = 1$ is an isocline which is a solution.

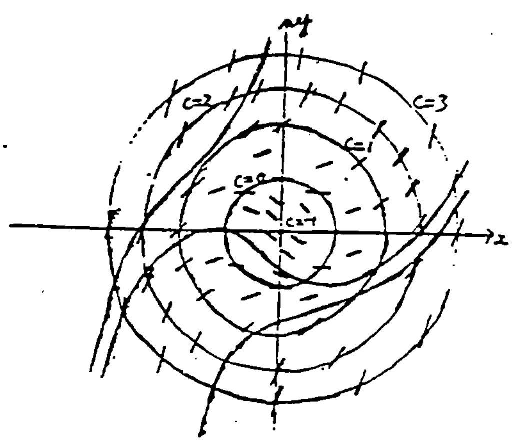


1C-1

d)

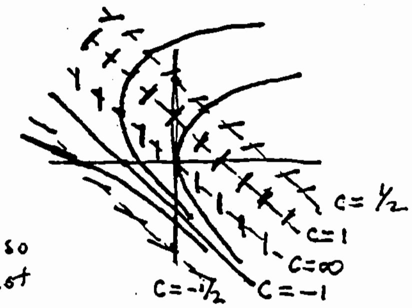
Isoclines: $x^2 + y^2 - 1 = c$

i.e. circles centre $(0,0)$, radius $\sqrt{1+c}$



1C-1

e) Isolines $x+y=1/c$ or $y=-x+\frac{1}{c}$



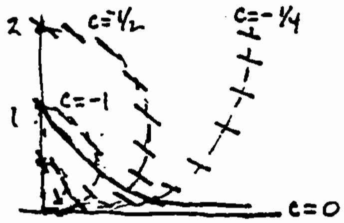
$y=-x-1$ is an integral curve, so other sol'ns cannot cross it.

1C-2

Isoclines: $x^2+y^2+\frac{y}{c}=0$, or completing the square:

$$x^2+\left(y+\frac{1}{2c}\right)^2=\left(\frac{1}{2c}\right)^2$$

(Circles, center at $(0,-1/2c)$.)



a) decreasing, since $y'=\dfrac{-y}{x^2+y^2}<0$ when $y>0$

b) soln must have $y>0$ for $x>0$ since it cannot cross the integral curve $y=0$.

1C-3

a) Using $\Delta y_n = hf(x_n,y_n) = h(x_n-y_n)$,

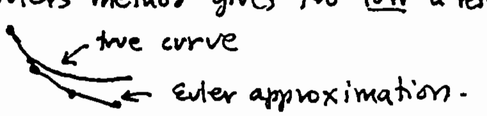
get $\quad y_{n+1} = y_n + h(x_n-y_n)$.

Table entries:

x	0	.1	.2	.3
y	1	.9	.82	.758

For example,
$$y_1 = y_0 + h(x_0-y_0) = 1 + .1(-1) = .9$$
$$y_2 = y_1 + h(x_1-y_1) = .9 + .1(.1-.9) = .82$$
$$y_3 = .82 + .1(.2-.82) = .758$$

some isolines $x-y=c$ are drawn.
sol'n curve through $(0,1)$ is convex (= "concave up");
thus Euler's method gives too <u>low</u> a result:



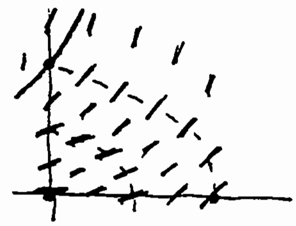
1C-4

Euler method formula:
$$y_{n+1} = y_n + hf(x_n,y_n)$$

x_n	y_n	$f(x_n,y_n)$	$hf(x_n,y_n)$
0	1	1	.1
.1	1.1	1.31	.131
.2	1.231	1.72	.172
.3	1.403		

$h=.1$

$f(x,y)=x+y^2$



isoclines $x+y^2=c$
(parabolas ⊃⊃⊃)

Solution curve through $(0,1)$ is convex ("concave up"),

∴ Euler method gives too <u>low</u> a result (same reasoning as in 19a)

1C-3

b) $\Delta y_n = \frac{h}{2}\left[f(x_n,y_n)+f(x_{n+1},\bar{y}_{n+1})\right]$

$y_{n+1}-y_n$

For this ODE, $f(x,y)=x-y$

($\bar{y}_{n+1}$ is the value given by the next step of Euler's method).

Thus.

So, $y_0 = 1$, $\bar{y}_1 = .9$ (from part a)

$$\therefore\ y_1-y_0 = \frac{.1}{2}\left[f(0,1)+f(.1,.9)\right] = \frac{.1}{2}\left[-1-.8\right] = -.09$$

$$\therefore\ y_1 = y_0 - .09$$
$$y_1 = 1-.09 = \boxed{.91}$$

This does correct the Euler value ($\bar{y}_1=.9$) in the right direction, since we predicted it would be too low. (.910 is actually the correct value of the sol'n to 3 places.)

1C-5

By the formula in 19a,

$$y_n = y_{n-1} + h(x_{n-1} - y_{n-1})$$
$$= (1-h)y_{n-1} + h x_{n-1}.$$

But for $x_0 = 0$, we get $x_1 = h$, $x_2 = 2h$, and in general $x_{n-1} = (n-1)h$.

$$\therefore \quad \boxed{y_n = (1-h)\,y_{n-1} + h^2(n-1)} \quad (**)$$

We prove by induction that the explicit formula for y_n is:

$$(*) \quad \boxed{y_n = 2(1-h)^n - 1 + nh}$$

a) it's true if $n=0$, since
$$y_0 = 2(1-h)^0 - 1 + 0 = 1 \quad ✓$$

b) if true for y_n, it's true for y_{n+1}: since, using $(*)$,
$$y_{n+1} = (1-h)y_n + h^2 n$$
$$= 2(1-h)^{n+1} + (1-h)(-1+nh) + h^2 n$$
$$\therefore \quad y_{n+1} = (1-h)^{n+1} - 1 + (n+1)h \quad ✓$$

[Note: $(**)$ is called a "difference equation" — there are standard ways to solve such things; here $(*)$ is the solution].

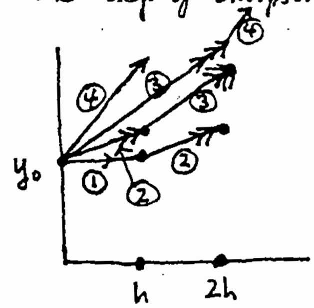
Continuing, in our case $h = 1/n$
$$\therefore \quad y_n = 2\left(1-\tfrac{1}{n}\right)^n - 1 + 1$$
$$= 2\left(1-\tfrac{1}{n}\right)^n.$$

$$\lim_{n\to\infty} y_n = 2e^{-1} \quad \left(\text{since } \lim_{k\to\infty}\left(1+\tfrac{1}{k}\right)^k = e;\ \text{put } k = -n\right)$$

The exact sol'n to the equation is
$$y = 2e^{-x} - 1 + x$$
so
$$y(1) = 2e^{-1} - 1 + 1 = 2e^{-1},$$
which checks.

1C-6

It suffices to prove this is true for one step of the Runge-Kutta method and one step of Simpson's rule.



We calculate, in R-K method, the 4 slopes marked 1 → 4

Then we use a weighted average of them to find $y(2h)$:

$$y_{2h} = y_0 + 2h\cdot\left(\frac{1 + 2\cdot 2 + 2\cdot 3 + 4}{6}\right)$$

Since the ODE is simply:
$$y' = f(x),$$
from the picture

slope 1 $= f(0)$
slope 2 $= f(h)$
slope 3 $= f(h)$
slope 4 $= f(2h)$

$$\therefore \quad y_{2h} = y_0 + \frac{2h}{6}\left(f(0) + 4f(h) + f(2h)\right)$$

Contrast this with the exact formula:
$$y_{2h} = y_0 + \int_0^{2h} f(x)\,dx$$

Evaluating the integral approximately by one step of Simpson's rule:
$$y_{2h} = y_0 + \frac{2h}{6}\left(f(0) + 4f(h) + f(2h)\right),$$
same as what Runge-Kutta gives.

1C-7

The existence and uniqueness theorem requires the equation to be written in the form

$$y' = f(x,y).$$

Doing this, we get

$$y' = -\frac{b(x)}{a(x)}\,y + \frac{c(x)}{a(x)}.$$

The conditions then are:

"$f(x,y)$ continuous" which will be so if

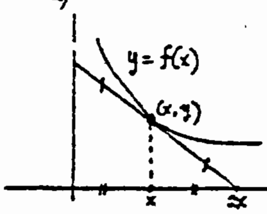
$\boxed{a(x),\ b(x),\ c(x) \text{ continuous (in an interval } [x_0-h, x_0+h]) \text{ and } a(x) \neq 0 \text{ in this interval.}}$

"$f_y(x,y)$ continuous", which will be so if $\frac{b(x)}{a(x)}$ is continuous, — and this is already implied by the above condition.

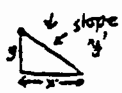
[Note that we must have $a(x) \neq 0$, a condition which is often missed.]

1D-1

a)



If (x,y) is a point on the curve, the geometric condition translates to:



$$\text{slope of tan. line} = \frac{y}{-x}$$

$$\therefore\ y' = -\frac{y}{x}$$

The solution (sep. of vars.) is $\boxed{y = \frac{c}{x}}$ [hyperbolas]

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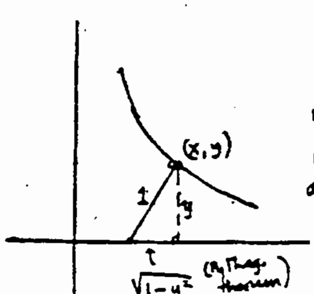
b)



Since the normal is $\perp$ to the tangent, its slope is the negative reciprocal.

$$\therefore\ \frac{y}{\sqrt{1-y^2}} = -\frac{1}{y'}$$

Solve by sep. of variables: $-\frac{y\,dy}{\sqrt{1-y^2}} = dx$

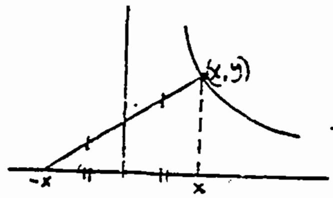
$\therefore\ \sqrt{1-y^2} = x + c$ or

$(x+c)^2 + y^2 = 1$ (Circles, radius 1, centre on x-axis — obvious, what?)

$y = \pm 1$ are also solutions to the problem (above assumed implicitly that $y \neq \pm 1$)

1D-1

(c)



Equating slopes of normal:

$$\frac{y}{2x} = -\frac{1}{y'}$$ (neg. recip. of slope of tangent)

Solve by sep. vars, get $\boxed{\frac{1}{2}y^2 + x^2 = c}$ (ellipses)

(d)

The required property translates mathematically into:

$$\int_a^x y(t)\,dt = k\big(y(x) - y(a)\big)$$

k = constant of proportionality

Differentiate this to get an ODE for $y(x)$:

$$y(x) = k\,y'(x)$$

(by 2nd Fund. Thm of Calculus)

solution: $\boxed{y = ce^{x/k}}$ This is the general exponential curve.

1D-2

(a)

(i) The y-intercept of line $y = mx + c$ is $(0, c)$ $\therefore\ c = 2m$

ie $y = mx + 2m = m(x+2)$

Differentiating $\Rightarrow y' = m$

Eliminate m: $\therefore\ \boxed{y' = \frac{y}{x+2}}$ ODE of family

(ii) Orthogonal trajectories satisfy: $-\frac{1}{y'} = \frac{y}{x+2}$

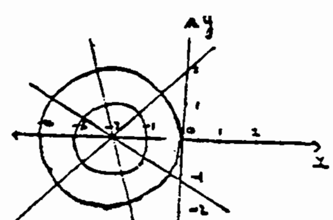
$\Rightarrow \frac{-dx}{dy} = y/x+2 \Rightarrow y\,dy = -x\,dx + 2\,dx$

$\therefore\ \frac{y^2}{2} + \frac{x^2}{2} + 2x = \text{constant}$

$\therefore\ (x+2)^2 + y^2 = k$

ie. Circles centre $(-2, 0)$, variable radii

(iii)



Original family Lines thro' $(-2,0)$

Orthogonal trajectories Circles centre $(-2,0)$

1D-2

(b)

$$y = ce^x$$

$$y' = ce^x = y$$

Equation of the orthogonal family:

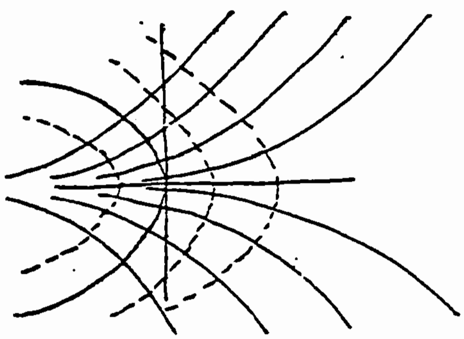
$$y' = -\frac{1}{y}$$

To find the curves, solve by separation of variables:

$$y\,dy = -dx$$

$$\tfrac{1}{2}y^2 = -x + c \qquad \underline{\text{parabolas}}$$

(all translations of one fixed parabola $\tfrac{1}{2}y^2 = -x$ along the x-axis)



1D-2

(c)

(i) Differentiating gives

$$2x - 2yy' = 0$$

$$\therefore \quad y' = \frac{x}{y} \quad \text{is required ODE}$$

(ii) Orthogonal trajectories satisfy

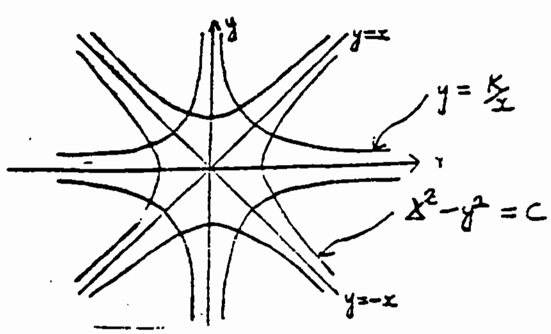
$$-\frac{1}{y'} = \frac{x}{y}$$

$$\therefore \quad \frac{-dy}{y} = \frac{dx}{x}$$

$$\therefore \quad -\ln y = \ln x + c_1$$

$$\therefore \quad y = \frac{K}{x}$$

(iii)



1D-2

(d) Circles with centre on y-axis have equation $x^2 + (y-k)^2 = r^2$

Circle tangent to x-axis $\Rightarrow r = \pm k \quad \therefore r^2 = k^2$

$$\therefore \quad x^2 + y^2 - 2yk = 0$$

$$\therefore \quad \frac{x^2+y^2}{2y} = k.$$

Differentiate w.r.t. x:

$$\therefore \quad \frac{2x + 2yy'}{2y} - \frac{(x^2+y^2)y'}{2y^2} = 0$$

$$\therefore \quad 2xy + 2y^2y' - x^2y' - y^2y' = 0$$

$$\text{i.e.} \quad y' = \frac{2xy}{x^2-y^2}$$

(ii) Orthogonal trajectories satisfy

$$-\frac{1}{y'} = \frac{2xy}{x^2-y^2}$$

$$\text{i.e.} \quad y' = \frac{y^2-x^2}{2xy} \quad \leftarrow \text{a } \underline{\text{homogeneous}} \text{ equation}$$

Let $y = zx \quad \therefore z = \frac{y}{x}$

Then $y' = xz' + z$

$$\therefore \quad xz' + z = \frac{z^2x^2 - x^2}{2zx^2} = \frac{z^2-1}{2z}$$

$$\therefore \quad xz' = \frac{-(z^2+1)}{2z} \quad \text{i.e.} \quad \frac{2z\,dz}{z^2+1} = \frac{-dx}{x}$$

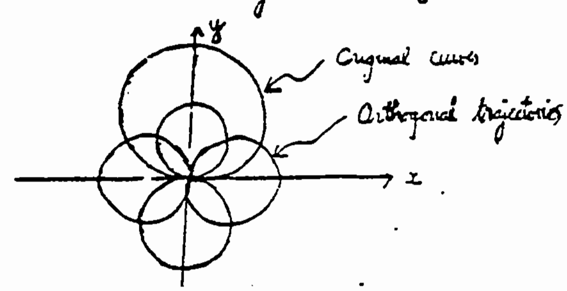
$$\therefore \quad \ln(z^2+1) = -\ln x + c$$

$$\therefore \quad z^2 + 1 = \frac{2K}{x} \qquad (2K = e^c)$$

$$\therefore \quad y^2 + x^2 = 2Kx$$

These are circles with centre on the x-axis and tangent to y-axis

(iii)



1D-3

a) $\frac{dx(t)}{dt}$ = rate at which salt flows in − rate of salt outflow

$= \begin{pmatrix}\text{flow}\\ \text{rate}\\ \text{in}\end{pmatrix}\cdot\begin{pmatrix}\text{conc.}\\ \text{salt}\\ \text{in added}\\ \text{sol'n}\end{pmatrix} - \begin{pmatrix}\text{flow}\\ \text{rate}\\ \text{out}\end{pmatrix}\cdot\begin{pmatrix}\text{conc.}\\ \text{of salt}\\ \text{in tank}\end{pmatrix}$

$$\boxed{x' = kc_1 - k\cdot\frac{x}{V}}$$

b) $x' + ax = 0$ (since $c_1 = 0$)
$x(0) = Vc_0$ $(a = k/V)$

Solution is, by sep. of variables
$x = Vc_0 e^{-at}$ $(a = k/V)$

c) The general case is $\begin{cases} x' + ax = kc_1, \\ x(0) = Vc_0 \end{cases}$ which can be solved by separating variables, or as a linear equation.

Separating variables:

$$\frac{dx}{dt} = kc_1 - ax$$

$$\frac{dx}{kc_1 - ax} = dt$$

$$-\frac{1}{a}\ln(kc_1 - ax) = t + A \quad \text{const. of integration}$$

or $kc_1 - ax = A_1 e^{-at}$ A_1 = arbitrary constant

Using the initial condition to find A_1: (note that $aV = k$)

$$kc_1 - aVc_0 = A_1$$

$$\therefore\ k(c_1 - c_0) = A_1$$

so soln is (note that $k/a = V$)

$$x = Vc_1 - V(c_1 - c_0)e^{-at}$$

or in terms of the concentration $c(t)$:

$$\boxed{c = c_1 - (c_1 - c_0)e^{-at}}$$

As $t \to \infty$, $e^{-at} \to 0$, so $\boxed{c \to c_1}$

d) If $c_1 = c_0 e^{-\alpha t}$, then the ODE (IVP) becomes (as in (c))

$$\begin{cases} x' + ax = kc_0 e^{-\alpha t} \\ x(0) = Vc_0 \end{cases}$$

This must be solved as a linear equation. The integrating factor is e^{at}:

$$x'e^{at} + axe^{at} = kc_0 e^{(a-\alpha)t}$$

or $(xe^{at})' = kc_0 e^{(a-\alpha)t}$ ⊛

Integrating,

$$xe^{at} = \frac{kc_0}{a-\alpha}e^{(a-\alpha)t} + A \quad \text{const. of integ.}$$

Using the initial condition to find A:

$$Vc_0 = A + \frac{kc_0}{a-\alpha}$$

$$\therefore\ x = \frac{kc_0}{a-\alpha}e^{-\alpha t} + \left(Vc_0 - \frac{kc_0}{a-\alpha}\right)e^{-at}$$

Dividing by V to get concentration:

$$c = \frac{ac_0}{a-\alpha}e^{-\alpha t} + \left(c_0 - \frac{ac_0}{a-\alpha}\right)e^{-at}$$

[If $\alpha = 0$, then $c_1 = c_0$, and this agrees with part (c)]

1D-4

$\frac{dA}{dt} = -\lambda_1 A$, $\lambda_1 = \frac{\ln 2}{\text{half-life}}$

$\frac{dB}{dt}$ = rate at which B produced by decay of A − rate at which B is lost by decay of B

$$\therefore\ \frac{dB}{dt} = \lambda_1 A - \lambda_2 B$$

$\therefore$, From the first equation, $A = A_0 e^{-\lambda_1 t}$

$$\therefore\ \boxed{\frac{dB}{dt} + \lambda_2 B = \lambda_1 A_0 e^{-\lambda_1 t}} \quad \text{ODE for } B(t)$$

Solve it as a linear equation, using $e^{+\lambda_2 t}$ as integrating factor, and $B(0) = B_0$ as initial condition. Solution is

$$B(t) = \frac{\lambda_1 A_0}{\lambda_2 - \lambda_1}e^{-\lambda_1 t} + \left(B_0 - \frac{\lambda_1 A_0}{\lambda_2 - \lambda_1}\right)e^{-\lambda_2 t}$$

Taking $\lambda_1 = 1$, $\lambda_2 = 2$,

$$B(t) = A_0 e^{-t} + (B_0 - A_0)e^{-2t}$$

Differentiating to see when $B(t)$ is maximum:

$$0 = B'(t) = -A_0 e^{-t} - 2(B_0 - A_0)e^{-2t}$$

Solving for t: $\frac{A_0}{2(A_0 - B_0)} = e^{-t}$

If $A_0 > 2B_0$, then $t = -\ln\left(\frac{A_0}{2(A_0 - B_0)}\right) > 0$

If $A_0 \le 2B_0$, no solution (the maximum is at $t = 0$).

$\boxed{1D-5}$

By Newtons cooling law

$$\frac{dT}{dt} = K(T-20)$$

(K a constant of proportionality)

Solving this (by sep. of variables) gives

$$T = \alpha e^{Kt} + 20 \qquad (\alpha \text{ another constant})$$

$$T(0) = 100$$

$$\therefore \quad \alpha + 20 = 100$$

$$\therefore \quad \alpha = 80$$

$$T(5) = \alpha e^{5K} + 20 = 80$$

$$\therefore \quad \alpha e^{5K} = 60$$

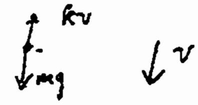
$$\therefore \quad K = \tfrac{1}{5}\ln\left(\tfrac{60}{\alpha}\right) = \tfrac{1}{5}\ln\left(\tfrac{3}{4}\right) < 0.$$

$$\therefore \quad T = 80\, e^{-\frac{1}{5}\ln(\frac{4}{3})t} + 20$$

When $T = 60$ we then find

$$t = \frac{5\ln 2}{\ln(\frac{4}{3})} \simeq 12 \text{ mins.}$$

$\boxed{1D-6}$

a) 

Downwards force $= m\frac{dv}{dt} = mg - kv$

$$\therefore \quad \frac{dv}{dt} + \frac{k}{m}v = g$$

Solving this by separation of variables (or as a linear equation), we get

$$v = \alpha e^{-k/m\,t} + \frac{mg}{k} \qquad (\alpha \text{ constant})$$

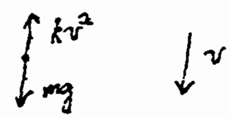
Using the initial condition

$$v(0) = 0. \quad \therefore \quad \frac{mg}{k} + \alpha = 0.$$

$$\therefore \quad v = \frac{mg}{k}\left(1 - e^{-kt/m}\right) \qquad \text{SOLN.}$$

terminal velocity:

$$\lim_{t\to\infty} v(t) = \frac{mg}{k} \qquad (\text{constant})$$

b) 

Downwards force $= m\frac{dv}{dt} = mg - kv^2$

$$\therefore \quad \frac{dv}{v^2 - \frac{mg}{k}} = -\frac{k}{m}\,dt$$

But $\dfrac{1}{v^2 - \frac{mg}{k}} = \dfrac{1}{v^2 - a^2} = \dfrac{1}{2a}\left[\dfrac{1}{v-a} - \dfrac{1}{v+a}\right]$

where $a \equiv \sqrt{\frac{mg}{k}}$

$$\therefore \quad \frac{dv}{v-a} - \frac{dv}{v+a} = -\frac{2ak}{m}\,dt$$

$$\therefore \quad \ln\left|\frac{v-a}{v+a}\right| = C - \frac{2ak}{m}t$$

But $v(0) = 0 \quad \therefore \ln 1 = C$
i.e., $C = 0$

$$\therefore \quad \frac{a-v}{a+v} = e^{-\frac{2akt}{m}}$$

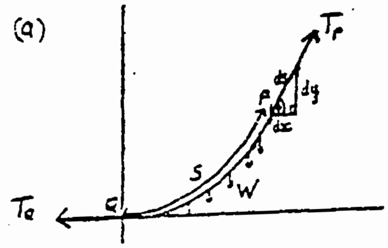
(since L.H.S. > 0 (at least near $t=0$))

$$\therefore \quad v = a\left(\frac{1 - e^{-\frac{2akt}{m}}}{1 + e^{-\frac{2akt}{m}}}\right)$$

$$\therefore \lim_{t\to\infty} v(t) = a = \sqrt{\frac{mg}{k}}$$

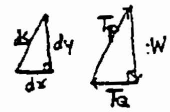
1D-7

(a)



$\tan\phi = \frac{dy}{dx}$

OR: the Δs are similar:



(Δ of forces is closed since cable is in equilibrium)

$\therefore$ $\frac{dx}{T_a} = \frac{dy}{W} = \frac{ds}{T_P}$

(corresponding sides)

Balancing forces horizontally

$$T_a = T_P \cos\phi = T_P \frac{dx}{ds}$$

$$\therefore \quad \frac{ds}{T_P} = \frac{dx}{T_a}. \qquad (i)$$

Balancing forces vertically

$$W = T_P \sin\phi = T_P \frac{dy}{ds}$$

$$\therefore \quad \frac{ds}{T_P} = \frac{dy}{W} \quad (ii) \quad \text{as required.}$$

(b) Suppose the cable hangs under its own weight and has constant density ρ per unit length

Then $W = \rho s$

Now $\frac{dx}{T_a} = \frac{dy}{W} = \frac{dy}{\rho s}$

$\therefore$ $\frac{dy}{dx} = ks$ (where $k = \frac{\rho}{T_a}$ is a constant)

Then $\frac{d^2y}{dx^2} = k\frac{ds}{dx} = \frac{k\sqrt{(dx)^2+(dy)^2}}{dx}$

$= k\sqrt{1+(y')^2}$ which proves (i).

Also, $\frac{dy}{W} = \frac{ds}{T_p}$; but $T_p = \sqrt{W^2+T_a^2}$ (from the force triangle)

$\therefore$ $\frac{dy}{\rho s} = \frac{ds}{\sqrt{\rho^2 s^2 + T_a^2}}$

$\therefore$ $\frac{dy}{ds} = \frac{s}{\sqrt{s^2+c^2}}$ where $c = T_a/\rho$

$\therefore$ $y = \sqrt{s^2+c^2} + c_1$,

which proves (ii)

(c) Let λ be the constant weight per unit horizontal length

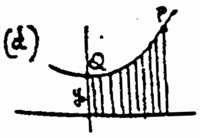
$$\therefore \quad W = \lambda x$$

Then $\frac{dy}{dx} = \frac{W}{T_a} = \frac{\lambda x}{T_a}$

$$\therefore \quad y = \frac{\lambda}{T_a}\frac{x^2}{2} + y_0$$

Thus the cable takes the form of a parabola.

(d)



Here $W = k_1 \cdot$(area under $\widehat{QP}$) since rods are equally and closely spaced.

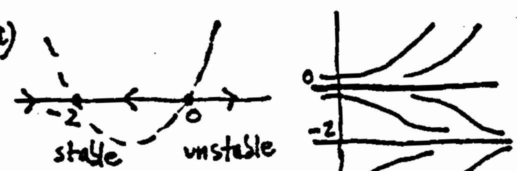
So $\frac{dy}{dx} = \frac{W}{T_a} = \frac{k_1}{T_a}\int_0^x y(t)\,dt$

$\therefore$ $\frac{d^2y}{dx^2} = k^2 y$, by the 2nd Fund. Thm. of Calculus. ($k^2 = k_1/T_a > 0$)

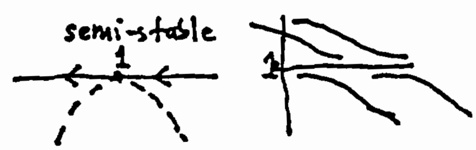
[The curve is once again of the form $y = \cosh(kx) + c_1$]

1E-1

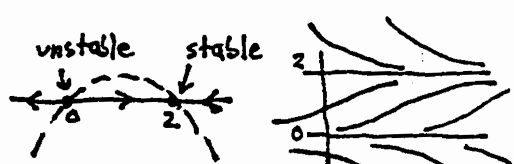
a)



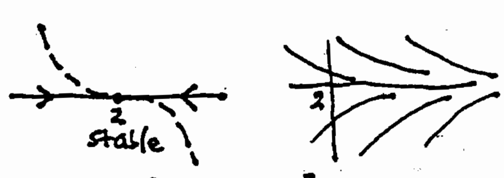
b)



c)



d)



(write: $(2-x)^3 = -(x-2)^3$)