4A-1 Product is $\begin{bmatrix}1 & 0 & 0\\ 3 & -2 & -6\end{bmatrix}$

4A-2 $AB = \begin{bmatrix}4 & 1\\ -2 & -4\end{bmatrix}$ $BA = \begin{bmatrix}-3 & 1\\ 5 & 3\end{bmatrix}$

4A-3 $A^{-1} = \frac{1}{-2}\cdot\begin{bmatrix}2 & -2\\ -3 & 2\end{bmatrix}$ by the formula

(since $|A| = -2$) $= \begin{bmatrix}-1 & 1\\ 3/2 & -1\end{bmatrix}$

check: $\begin{bmatrix}2 & 2\\ 3 & 2\end{bmatrix}\begin{bmatrix}-1 & 1\\ 3/2 & -1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

4A-4 $\frac{1}{|A|}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}\cdot\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

$= \frac{1}{|A|}\cdot\begin{bmatrix}ad-bc & 0\\ 0 & ad-bc\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

(similarly, $\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix} = \begin{bmatrix}|A| & 0\\ 0 & |A|\end{bmatrix}$)

4A-5 $\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix} = A^2$

$\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix} = \begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$

4A-6 Using determinantal criterion for lin. dependence,

we want $0 = \begin{vmatrix}1 & 2 & c\\ -1 & 0 & 1\\ 2 & 3 & 0\end{vmatrix} = 4 - 3c - 3$

$\therefore -3c + 1 = 0$

$c = 1/3$

Adding: $(1\ \ 2\ \ \overset{1/3}{\overset{\shortparallel}{c}}) \times 3$

$-\ (-1\ \ 0\ \ 1)$

$-\ (2\ \ 3\ \ 0) \times 2$

$(0\ \ 0\ \ 0)$

4B-1 a) $x'' + 5x' + tx^2 = 0 \rightarrow \begin{matrix}x' = y\\ y' = -tx^2 - 5y\end{matrix}$

b) $y'' - x^2y' + (1-x^2)y = \sin x$

$\rightarrow \begin{matrix}y' = z\\ z' = (x^2-1)y + x^2 z + \sin x\end{matrix}$

4B-2

$y''' + py'' + qy' + ry = 0$

let $y = y_1$

$y_1' = y_2$

$y_2' = y_3$

$y_3' = -py_3 - qy_2 - ry_1$

matrix form: $\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}' = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -r & -q & -p\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}$

4B-3

$\begin{cases}x' = x + y\\ y' = 4x + y\end{cases}$ To eliminate y: $y = x' - x$ from 1st eq'n.

$\therefore (x'-x)' = 4x + (x'-x)$ 2nd eq.

or $x'' - x' = 4x + x' - x$

or $\boxed{x'' - 2x' - 3x = 0}$

converting to system:

let $x_1 = x$

system is $\begin{cases}x_1' = x_2\\ x_2' = 2x_2 + 3x_1\end{cases}$ → This system is not same as first, but is equivalent to it — just using different dep't variables.

The rel'n between the variables is:

$x_1 = x$

$x_2 = x + y$ or the other way: $\left[\begin{matrix}x = x_1\\ y = x_2 - x_1\end{matrix}\right.$

If you make this change of vars. the 1st system turns into the second.

4B-4

$\begin{bmatrix}1\\ 1\end{bmatrix}e^{3t}$ and $\begin{bmatrix}1\\ 2\end{bmatrix}e^{2t}$ solve $\vec{x}' = \begin{bmatrix}4 & -1\\ 2 & 1\end{bmatrix}\vec{x}$:

a) vectorially: $\frac{d}{dt}\begin{bmatrix}1\\ 1\end{bmatrix}e^{3t} = 3\begin{bmatrix}1\\ 1\end{bmatrix}e^{3t}$ these are equal. Other goes same way.

$\begin{bmatrix}4 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}e^{3t} = \begin{bmatrix}3\\ 3\end{bmatrix}e^{3t}$

components: $\begin{matrix}x = e^{3t}\\ y = e^{3t}\end{matrix}$ solves $\begin{cases}x' = 4x - y\\ y' = 2x + y\end{cases}$: just plug in & check it.

b) linearly indep't: $\begin{vmatrix}e^{3t} & e^{2t}\\ e^{3t} & 2e^{2t}\end{vmatrix} = e^{5t} \neq 0$.

c) gen soln: $c_1\begin{bmatrix}1\\ 1\end{bmatrix}e^{3t} + c_2\begin{bmatrix}1\\ 2\end{bmatrix}e^{2t}$ or $\begin{bmatrix}c_1e^{3t} + c_2e^{2t}\\ c_1e^{3t} + 2c_2e^{2t}\end{bmatrix}$

which is same as: $x = c_1e^{3t} + c_2e^{2t}$

$y = c_1e^{3t} + 2c_2e^{2t}$.

4B-5 $\begin{bmatrix}1\\1\end{bmatrix}e^{4t}$ and $\begin{bmatrix}1\\-1\end{bmatrix}e^{-2t}$ solve $\begin{bmatrix}x\\y\end{bmatrix}' = \begin{bmatrix}1&3\\3&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$.

(do it same as N3 1a above). Linear indep: $\begin{vmatrix}e^{4t} & e^{-2t}\\ e^{4t} & -e^{-2t}\end{vmatrix} = -2e^{2t}$

IVP: $\vec{x}(0) = \begin{bmatrix}5\\1\end{bmatrix}$, gives: (since $e^{4t}, e^{-2t} = 1$ when $t=0$)

$$c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}5\\1\end{bmatrix} \quad \therefore \quad \begin{matrix}c_1+c_2=5\\ c_1-c_2=1\end{matrix} \quad \therefore \quad \begin{matrix}c_1=3\\ c_2=2\end{matrix}$$

soln: $3\begin{bmatrix}1\\1\end{bmatrix}e^{4t} + 2\begin{bmatrix}1\\-1\end{bmatrix}e^{-2t} = \vec{x}$.

4B-6 $\begin{bmatrix}x\\y\end{bmatrix}' = \begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$. or $\begin{matrix}x' = x+y\\ y' = y\end{matrix}$

a) From second eqn, $y = c_1e^t$

$\therefore\ x' - x = c_1e^t$ soln: $x = c_2e^t + c_1te^t$, $y = c_1e^t$

b) Here we eliminate y instead:

$y = x' - x$ (1st eqn.) $\therefore\ (x'-x)' = x' - x$

$x'' - 2x' + x = 0$ $\therefore\ x = c_1e^t + c_2te^t$

$(m-1)^2 = 0$ $\therefore\ y = c_2e^t$ (since $y = x'-x$)

Same as before (just switch c_1 & c_2).

4B-7 $x' = -ax$ (straight decay)

$y' = \underset{\text{decay rate}}{-by} + \underset{\text{rate at which decay of x produces y}}{ax}$.

matrix: $\begin{bmatrix}x\\y\end{bmatrix}' = \begin{bmatrix}-a&0\\a&-b\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

Sol'n by elimination: eliminate x : $x = \frac{1}{a}y' + \frac{b}{a}y$

subst. into 1st eqn, get

$$\frac{1}{a}y'' + \frac{b}{a}y' = -y' - by$$

$y'' + (b+a)y' + bay = 0$ $\quad m^2 + (a+b)m + ab = 0$

$\therefore\ y = c_1e^{-at} + c_2e^{-bt}$ $\quad m=-a,\ m=-b$

$x = c_1\left(-1+\frac{b}{a}\right)e^{-at}$ $\leftarrow$ $\left[\ x = \frac{1}{a}(y'+by) = \frac{1}{a}\left(-ac_1e^{-at} - bc_2e^{-bt} + bc_1e^{-at} + bc_2e^{-bt}\right)\right.$

[NOTE: having found y, you can't just say $x' = -ax$, $\therefore\ x = c_3e^{-at}$, since c_3 is not arbitrary — x must also satisfy the 2nd eqn !!

4C-1 a)

a) $\vec{x}' = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix} \vec{x}$ eigenvalues: $\begin{vmatrix} -3-m & 4 \\ -2 & 3-m \end{vmatrix} = 0$

$\therefore -(3+m)(3-m) + 8 = 0$

$m^2 - 1 = 0 \qquad m = \pm 1$

Eigenvectors:

if $m = 1$,

$\begin{cases} -4\alpha_1 + 4\alpha_2 = 0 \\ -2\alpha_1 + 2\alpha_2 = 0 \end{cases}$

$\therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and its mult. are soln; eigenvector.

if $m = -1$:

$-2\alpha_1 + 4\alpha_2 = 0$
$-2\alpha_1 + 4\alpha_2 = 0$ soln: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ eigenvector.

$\therefore \vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$

[NOTE: can also write down char. polyn. using:
$m^2 - (a_1 + b_2)m + \det A = 0$.

b) $\vec{x}' = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \vec{x}$ $\begin{vmatrix} 4-m & -3 \\ 8 & -6-m \end{vmatrix} = 0$ gives

$m^2 + 2m = 0 \qquad m = -2, \; m = 0$.

$m = 0$:
$4\alpha_1 - 3\alpha_2 = 0$
$8\alpha_1 - 6\alpha_2 = 0$ $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ eigenvector.

$m = -2$
$6\alpha_1 - 3\alpha_2 = 0$
$8\alpha_1 - 4\alpha_2 = 0$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ eigenv.

$\therefore \vec{x} = c_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-2t}$

4C-1

c) eigenvalues: $\begin{vmatrix} 1-m & -1 & 0 \\ 1 & 2-m & 1 \\ -2 & 1 & -1-m \end{vmatrix} = -(1-m)(2-m)(1+m) + 2 + (m-1) - 1 - m = 0$

$\boxed{(1-m)(2-m)(1+m) = 0}$

eigenvalues $\therefore$ are $m = 1, \; m = 2, \; m = -1$

$\underline{m = 1}$
$0\alpha_1 - \alpha_2 = 0$
$\alpha_1 + \alpha_2 + \alpha_3 = 0$
$-2\alpha_1 + \alpha_2 - 2\alpha_3 = 0$
$\therefore \alpha_2 = 0 \quad \alpha_3 = -\alpha_1$
soln: $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ eigenvector

$\underline{m = 2}$
$-\alpha_1 - \alpha_2 = 0$
$\alpha_1 + \alpha_3 = 0$
$-2\alpha_1 + \alpha_2 - 3\alpha_3 = 0$
$\therefore \alpha_2 = -\alpha_1$
$\alpha_3 = -\alpha_1$
$\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ eigenvector

$\underline{m = -1}$
$+2\alpha_1 - \alpha_2 = 0$
$\alpha_1 + 3\alpha_2 + \alpha_3 = 0$
$-2\alpha_1 + \alpha_2 = 0$
$\alpha_2 = +2\alpha_1$
$\alpha_3 = -7\alpha_1$
$\begin{bmatrix} 1 \\ +2 \\ -7 \end{bmatrix}$ eigenvector.

soln:
$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix} e^{-t}$

4C-2

Proof #1: $\therefore$ 0 is an eigenvalue if and only if $A\vec{\alpha} = 0\vec{\alpha}$ has a nontriv. soln for $\vec{\alpha}$

$\Leftrightarrow A\vec{\alpha} = \vec{0}$ " " " " "

$\Leftrightarrow \det A = 0$ (see notes p. 2 (5)).

Proof #2: The characteristic equation is $\det(A - mI) = 0$.
If $m = 0$ is a root, this says (substituting $m = 0$) $\det(A) = 0$

4C-3

$\begin{vmatrix} a-m & * & * \\ 0 & b-m & * \\ 0 & 0 & c-m \end{vmatrix} = (a-m)(b-m)(c-m) = 0$
$\therefore m = a, b, c$ are eigenvalues

This always holds: using a Laplace expansion by the minors of first column:

$\begin{vmatrix} a_1-m & * & \cdots & * \\ 0 & a_2-m & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & a_k-m \end{vmatrix} = (a_1-m) \begin{vmatrix} a_2-m & * & \cdots & * \\ 0 & \ddots & & * \\ 0 & \cdots & 0 & a_k-m \end{vmatrix}$

$= (a_1-m)(a_2-m)\cdots(a_k-m)$ by mathematical induction on the size of matrix (i.e., k)

$\therefore$ eigenvalues are the roots: $m = a_1, a_2, \ldots, a_k$ = diagonal elements.

4C-4

By hypothesis, $A\vec{\alpha} = m\vec{\alpha}$.
Multiply both sides by A:
$AA\vec{\alpha} = mA\vec{\alpha} = m(m\vec{\alpha})$.
$\therefore A^2\vec{\alpha} = m^2\vec{\alpha}$ so $\vec{\alpha}$ is eigenv. of A^2, assoc. to eigenvalue m^2.

[Continuing, one sees that $A^k\vec{\alpha} = m^k\vec{\alpha}$ — the eigenvalues of A^k are the kth powers of the eigenvalues of A].

4C-5

$\vec{x}' = \begin{bmatrix} -a & 0 \\ a & -b \end{bmatrix} \vec{x}$ Eigenvalues: $-a, -b$ (by previous problem, or directly)

$\underline{m = -a}$
$a\alpha_1 + (b+a)\alpha_2 = 0$
soln: $\begin{bmatrix} \frac{a-b}{a} \\ 1 \end{bmatrix}$ eigenvector

$\underline{m = -b}$
$(-a-b)\alpha_1 = 0$
$a\alpha_1 + 0\alpha_2 = 0$ $\therefore \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ soln. eigenvector.

$\therefore \vec{x} = c_1 \begin{bmatrix} \frac{a-b}{a} \\ 1 \end{bmatrix} e^{-at} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-bt}$

when written out with components, this is identical to our earlier solution...

4C-6

$S' = \underbrace{S}_{\text{natural growth rate}} - \underbrace{aS}_{\text{fraction that leave for Jones}} + \underbrace{bJ}_{\text{rabbits coming in from Jones}}$

$\therefore S' = (1-a)S + bJ$

$J' = aS + (1-b)J.$

$J' = J - bJ + aS$

if $a = b = 1/2$,

$$\begin{bmatrix} S \\ J \end{bmatrix}' = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} S \\ J \end{bmatrix}$$

Eigenvalues: $\begin{vmatrix} \frac12 - m & \frac12 \\ \frac12 & \frac12 - m \end{vmatrix} = m^2 - m = 0$

$m = 0,\ m = 1$ eigenvalues:

$\underline{m=0}$: $\frac12\alpha_1 + \frac12\alpha_2 = 0$: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad$ $\underline{m=1}$: $-\frac12\alpha_1 + \frac12\alpha_2 = 0$, $\frac12\alpha_1 - \frac12\alpha_2 = 0$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore \vec{x} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^t$ $\qquad$ IVP: $c_1 + c_2 = 20$, $-c_1 + c_2 = 10$ ← since $t=0$; $\quad c_2 = 15$, $c_1 = 5$

Sol'n: $\therefore \begin{bmatrix} S \\ J \end{bmatrix} = 5\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 15\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^t$

Nonoscillation: Show this char. polynomial never has complex roots: (Carry this part should come later!!) $m^2 + (a+b-2)m + (1-a-b) = 0$

4C-7

From the "picture":

$\frac14(x_1' - x_1) = x_2$

$x_2' - x_2 = x_1$

$\therefore \begin{cases} x_1' = x_1 + 4x_2 \\ x_2' = x_1 + x_2 \end{cases}$

solving:

eigenvalues: $\begin{vmatrix} 1-m & 4 \\ 1 & 1-m \end{vmatrix} = (1-m)^2 - 4 = 0$ $\quad\therefore 1 - m = \pm 2$ $\quad\therefore m = 3, -1$

$\underline{m=3}$: $-2\alpha_1 + 4\alpha_2 = 0$ soln: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\qquad$ $\underline{m=-1}$: $2\alpha_1 + 4\alpha_2 = 0$ soln: $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\therefore \vec{x} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{3t} + c_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{-t}$

Initial condition: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $\qquad$ $2c_1 - 2c_2 = 1$, $c_1 + c_2 = 0$

Soln: $\vec{x} = \frac14\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{3t} - \frac14\begin{bmatrix} -2 \\ 1 \end{bmatrix}e^{-t}$ $\quad\therefore c_1 = 1/4 \quad c_2 = -1/4$

4D-1

26. Characteristic equation: $m^2 + 4$; $\quad m = 2i$ eigenvalue

Corresponding eigenvector:

$\begin{cases} (1-2i)\alpha_1 - 5\alpha_2 = 0 \\ \alpha_1 + (-1-2i)\alpha_2 = 0 \end{cases}$ these are multiples of each other

Possible choices for eigenvector: $\begin{bmatrix} 5 \\ 1-2i \end{bmatrix}$ or $\begin{bmatrix} 1+2i \\ 1 \end{bmatrix}$

The second choice gives as the soln $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}i\right)(\cos 2t + i\sin 2t)$

with real part $\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin 2t$, imag part: $\begin{bmatrix} 2 \\ 0 \end{bmatrix}\cos 2t + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin 2t$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = c_1\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin 2t\right) + c_2\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\cos 2t + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin 2t\right)$

$\therefore x = (c_1 + 2c_2)\cos 2t + (c_2 - 2c_1)\sin 2t$

$y = c_1\cos 2t + c_2\sin 2t$

The other choice of eigenvector leads to $x = 5a_1\cos 2t + 5a_2\sin 2t$, $y = (a_1 - 2a_2)\cos 2t + (2a_1 + a_2)\sin 2t$

(an equivalent solution).

4D-2

Characteristic equation is $m^2 - 6m + 25 = 0$

$\therefore m = 3 \pm 4i$, by quadratic formula

Using $3+4i$ as complex eigenvalue, corresponding eigenvector comes from equation $(3-m)\alpha_1 + 4\alpha_2 = 0$ $\quad\therefore \vec{\alpha} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

Corresponding solution is formed from real + imag. parts of

$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}i\right)e^{3t}(\cos 4t + i\sin 4t)$, giving

$x = e^{3t}(c_1\cos 4t + c_2\sin 4t)$

$y = e^{3t}(c_1\sin 4t - c_2\cos 4t)$

4D-3

Char. equation is $(m-2)^2(m+1) = 0$

eigenvalue -1 gives eq'ns $3\alpha_1 + 3\alpha_2 + 3\alpha_3 = 0$, $-3\alpha_3 = 0$, $3\alpha_3 = 0$ $\quad\vec{\alpha} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

repeated eigenvalue 2 gives eq'ns $\begin{cases} 3\alpha_2 + 3\alpha_3 = 0 \\ -3\alpha_2 - 3\alpha_3 = 0 \\ 0 = 0 \end{cases}$ which have 2 lin. indep't solns, namely $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ — thus 2 is a complete eigenvalue

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}e^{-t} + c_2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}e^{2t} + c_3\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}e^{2t}$

or $\quad x = c_1e^{-t} + c_2e^{+2t}$

$y = -c_1e^{-t} + c_3e^{+2t}$

$z = -c_3e^{+2t}$

4D-4

a) $\underbrace{A_1'}_{\text{rate of change of salt in cell 1}} = \underbrace{(A_2 - A_1)}_{\text{rate of diffusion from } 2\to1} + \underbrace{(A_3 - A_1)}_{\text{rate of diffusion from } 3\to1}$ $\qquad A_2 - A_1 = x_2 - x_1$, $A_1' = x_1'$

$\therefore x_1' = x_2 - x_1 + x_3 - x_1 = -2x_1 + x_2 + x_3$

Similarly, $x_2' = x_1 - 2x_2 + x_3$

$x_3' = x_1 + x_2 - 2x_3$

b) Characteristic eq'n is $m^3 + 6m^2 + 9m = 0$

$= m(m+3)^2$

Eigenvalue 0 gives eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, normal mode is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

($e^{0t} = 1$, notice)

Eigenvalue -3 gives for eigenvector equations just

$\alpha_1 + \alpha_2 + \alpha_3 = 0$ (all 3 eq'ns are same)

This is a complete eigenvalue: it has multiplicity 2 and 2 lin indep sol'ns: $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

normal modes: $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}e^{-3t}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}e^{-3t}$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$: all 3 cells have same amt of salt — stays that way

$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}e^{-3t}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}e^{-3t}$ — one cell is at starting concentration A_0 + stays that way; other two cells are equally above + below A_0 at start; salt flows from one to other until "at ∞" they all have A_0 salt in them.

4E-1 $\vec{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$ Solving to get eigenvectors:

$\lambda^2 - 3\lambda - 10 = 0$ $\lambda = 5$ gives $\binom{2}{1}$

$(\lambda-5)(\lambda+2) = 0$ $\lambda = -2$ gives $\binom{1}{-3}$

[eq'ns are: $-a_1 + 2a_2 = 0$ and $6a_1 + 2a_2 = 0$, respectively]

$\therefore$ coörd. change is:

$$\binom{x}{y} = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}\binom{u}{v}$$

[can multiply each column by a constant and it's still OK]

Check it decouples: $x = 2u + v$, $y = u - 3v$

$\therefore$ substituting into system:

$$2u' + v' = 4(2u+v) + 2(u-3v) = 10 - 2v$$

$$u' - 3v' = 5u + 6v, \text{ similarly}$$

Multiply {top eqn by 3 and add; bot. eqn by 2 and subtract}

and you get $u' = 5u$, $v' = -2v$ decoupled!

4E-2 $\vec{x}' = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ use the eigenvectors given in 4D-4:

variable change matrix is:

$E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}$; $\vec{x} = E\vec{u}$ is the change of vars.

(cols are eigenvectors)

To check, use matrices: $\vec{u} = E^{-1}\vec{x}$ + subst. into system

$\boxed{\vec{u}' = E^{-1}AE\vec{u}}$

is the new system. Calculating:

$$\vec{u}' = \underbrace{\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{bmatrix}}_{E^{-1}}\underbrace{\begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & -3 \\ 0 & 3 & 3 \end{bmatrix}}_{A\cdot E}\vec{u}$$

$$\vec{u}' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}\vec{u}$$

So system is decoupled: $u_1' = 0$, $u_2' = -3u_2$, $u_3' = -3u_3$

4F-1 $x'' + px' + qx = 0$

a) $x' = y$ $\quad \vec{x} = \binom{x}{y} = \binom{x}{x'}$

b) $y' = -qx - py$

$\therefore$ Wronskian of two solutions $\vec{x}_1$ and $\vec{x}_2$ is $\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$, or $\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix}$, since $y_i = x_i'$, which is the usual Wronskian of x_1 and x_2.

4F-2

a) Neither is a constant multiple of the other.

b) $W(\vec{x}_1, \vec{x}_2) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = t^2$

c) Since $W = 0$ when $t = 0$, $\vec{x}_1$ and $\vec{x}_2$ cannot be solutions of $\vec{x}' = A(t)\vec{x}$, where the entries of $A(t)$ are continuous.

d) To find $A(t)$ explicitly, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $\boxed{\vec{x}' = A\vec{x}}$

Then since $\begin{bmatrix} t \\ 1 \end{bmatrix}$ is soln, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} t \\ 1 \end{bmatrix}$ or: $1 = at + b$, $0 = ct + d$

Since $\begin{bmatrix} t^2 \\ 2t \end{bmatrix}$ is soln, $\begin{bmatrix} 2t \\ 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} t^2 \\ 2t \end{bmatrix}$ or: $2t = at^2 + b2t$, $2 = ct^2 + d2t$

These are 4 equations for a, b, c, d. Solving: $a = 0$, $b = 1$, $c = -2/t^2$ $d = 2/t$ so not contin. at $t=0$

4F-3

a) $\begin{vmatrix} \alpha_1 e^{m_1 t} & \alpha_2 e^{m_2 t} \\ \beta_1 e^{m_1 t} & \beta_2 e^{m_2 t} \end{vmatrix} = e^{(m_1+m_2)t}\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix}$

$\vec{\alpha}_1 = \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}$ $\quad = 0 \Leftrightarrow \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = 0$

$\vec{\alpha}_2 = \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}$ $\quad \Leftrightarrow \vec{\alpha}_1, \vec{\alpha}_2$ are lin. dep't

b) Suppose $c_1\vec{\alpha}_1 + c_2\vec{\alpha}_2 = \vec{0}$ Multiply by A:

$$c_1 A\vec{\alpha}_1 + c_2 A\vec{\alpha}_2 = A\vec{0}$$

$$\therefore c_1 m_1\vec{\alpha}_1 + c_2 m_2\vec{\alpha}_2 = \vec{0}$$

Multiply top eq'n by m_1, subtract from 3^{rd} eq'n, get

$$c_2(m_2 - m_1)\vec{\alpha}_2 = \vec{0}$$

But $m_1 \neq m_2$, $\vec{\alpha}_2 \neq \vec{0}$ (since it's an eigenvector)

$\therefore$ $c_2 = 0$

$\therefore$ also $c_1 = 0$ (since $c_1\vec{\alpha}_1 = \vec{0}$ & $\vec{\alpha}_1 \neq \vec{0}$)

4F-4

If $\vec{x}'(0) = \vec{0}$, then since $\vec{x}' = A\vec{x}$, it follows that $A\,\vec{x}(0) = \vec{0}$, also.

Since A is nonsingular, we can multiply by A^{-1}, + get $\vec{x}(0) = \vec{0}$.

$\therefore$ by the uniqueness theorem, $\vec{x}(t) = \vec{0}$ for all t.

Hypotheses needed: A can be a function of t (with continuous entries); require only that at time $t=0$, $A(0)$ is nonsingular — then above reasoning still applies.

4G-1

a) Gen soln is: $\vec{x} = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{3t} + c_2\begin{bmatrix}1\\2\end{bmatrix}e^{2t}$

$\vec{x}_2(0) = \begin{bmatrix}0\\1\end{bmatrix}$: $\begin{bmatrix}0\\1\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\2\end{bmatrix}$

or: $c_1 + c_2 = 0$, $c_1 + 2c_2 = 1$ $\quad\therefore c_2 = 1,\ c_1 = -1$

$\therefore \vec{x}_2 = -\begin{bmatrix}1\\1\end{bmatrix}e^{3t} + \begin{bmatrix}1\\2\end{bmatrix}e^{2t}$: solves $\vec{x}_2(0) = \begin{bmatrix}0\\1\end{bmatrix}$

b) $\vec{x}_1 = 2\begin{bmatrix}1\\1\end{bmatrix}e^{3t} - \begin{bmatrix}1\\2\end{bmatrix}e^{2t}$ solves $\vec{x}_1(0) = \begin{bmatrix}1\\0\end{bmatrix}$

$\therefore$ soln to $\vec{x}(0) = \begin{bmatrix}a\\b\end{bmatrix}$ is: $\boxed{a\vec{x}_1 + b\vec{x}_2}$

(since $\begin{bmatrix}a\\b\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}$)

$$a\vec{x}_1 + b\vec{x}_2 = (2a-b)\begin{bmatrix}1\\1\end{bmatrix}e^{3t} + (b-a)\begin{bmatrix}1\\2\end{bmatrix}e^{2t}$$

4G-2

a) $\vec{x}' = \begin{bmatrix}5 & -1\\3 & 1\end{bmatrix}\vec{x}$ Eigenvalues: $\begin{vmatrix}5-m & -1\\3 & 1-m\end{vmatrix} = m^2 - 6m + 8 = 0$, $m = 4, 2$

$m=4$: $\alpha_1 - \alpha_2 = 0$; $\begin{bmatrix}1\\1\end{bmatrix}$ eigenvector, $\begin{bmatrix}1\\1\end{bmatrix}e^{4t}$ soln

$m=2$: $3\alpha_1 - \alpha_2 = 0$; $\begin{bmatrix}1\\3\end{bmatrix}$ eigvector, $\begin{bmatrix}1\\3\end{bmatrix}e^{2t}$ solution.

Fund. matrix: $\begin{bmatrix}e^{2t} & e^{4t}\\3e^{2t} & e^{4t}\end{bmatrix} = F(t)$ $\quad F(0) = \begin{bmatrix}1 & 1\\3 & 1\end{bmatrix}$ $\quad F(0)^{-1} = -\frac{1}{2}\begin{bmatrix}1 & -1\\-3 & 1\end{bmatrix}$

Soln to IVP:

$$= F(t)F(0)^{-1}\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}e^{2t} & e^{4t}\\3e^{2t} & e^{4t}\end{bmatrix}\begin{bmatrix}-1/2 & 1/2\\3/2 & -1/2\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}$$

[see p.12, (34)]

$$= \begin{bmatrix}e^{2t} & e^{4t}\\3e^{2t} & e^{4t}\end{bmatrix}\begin{bmatrix}-3/2\\7/2\end{bmatrix} = -\frac{3}{2}\begin{bmatrix}e^{2t}\\3e^{2t}\end{bmatrix} + \frac{7}{2}\begin{bmatrix}e^{4t}\\e^{4t}\end{bmatrix}$$

b) Normalized fund. mx: $\begin{bmatrix}e^{2t} & e^{4t}\\3e^{2t} & e^{4t}\end{bmatrix}\begin{bmatrix}-1/2 & 1/2\\3/2 & -1/2\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}e^{2t}+\frac{3}{2}e^{4t} & \frac{1}{2}e^{2t}-\frac{1}{2}e^{4t}\\-\frac{3}{2}e^{2t}+\frac{3}{2}e^{4t} & \frac{3}{2}e^{2t}-\frac{1}{2}e^{4t}\end{bmatrix}$

Multiply this on right by $\begin{bmatrix}2\\-1\end{bmatrix}$ to get same answer.

4H-1

$A = \begin{bmatrix}a & 0\\0 & b\end{bmatrix}$, $A^2 = \begin{bmatrix}a^2 & 0\\0 & b^2\end{bmatrix}$, $\dots$ $A^n = \begin{bmatrix}a^n & 0\\0 & b^n\end{bmatrix}$ by rules for mx. mult.

$$\therefore e^{At} = I + At + A^2\frac{t^2}{2!} + \dots$$
$$= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}at & 0\\0 & bt\end{bmatrix} + \begin{bmatrix}\frac{a^2t^2}{2!} & 0\\0 & \frac{b^2t^2}{2!}\end{bmatrix} + \dots$$
$$= \begin{bmatrix}1 + at + \frac{a^2t^2}{2!} + \dots & 0\\0 & 1 + bt + \frac{b^2t^2}{2!} + \dots\end{bmatrix} = \begin{bmatrix}e^{at} & 0\\0 & e^{bt}\end{bmatrix}$$

$$\vec{x} = e^{At}\vec{x}_0 = \begin{bmatrix}e^{at} & 0\\0 & e^{bt}\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix} = \begin{bmatrix}k_1e^{at}\\k_2e^{bt}\end{bmatrix}$$

verify: $x = k_1e^{at}$, $y = k_2e^{bt}$ is soln of: $\begin{cases}x' = ax\\y' = by\end{cases}$ with $x(0) = k_1$, $y(0) = k_2$ obvious!

4H-2

$A = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$ $\quad A^2 = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$, $A^3 = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$, $A^4 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

after this it repeats (since $A^4 = I$)

ie, $A^5 = A$, $A^6 = A^2$, etc.

$$e^{At} = \begin{bmatrix}1 - t^2/2! + t^4/4! \cdots & t - t^3/3! + t^5/5! - \cdots\\-t + t^3/3! \cdots & 1 - t^2/2! + t^4/4! \cdots\end{bmatrix}$$
$$= \begin{bmatrix}\cos t & \sin t\\-\sin t & \cos t\end{bmatrix}$$

$$\vec{x} = e^{At}\begin{bmatrix}k_1\\k_2\end{bmatrix} = \begin{bmatrix}\cos t & \sin t\\-\sin t & \cos t\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix} = \begin{bmatrix}k_1\cos t + k_2\sin t\\-k_1\sin t + k_2\cos t\end{bmatrix}$$

This obviously satisfies the system: (I.V.P.) $x' = y$, $y' = -x$, $x(0) = k_1$, $y(0) = k_2$.

4H-4

$$e^{At} = I + At + A^2\frac{t^2}{2!} + \dots \quad (*)$$

In general, for matrices B, C, (square),

$$\frac{d}{dt}B(t)C(t) = \frac{dB}{dt}C + B\frac{dC}{dt}$$

$$\therefore \frac{d}{dt}A(t)A(t) = \frac{dA}{dt}\cdot A + A\cdot\frac{dA}{dt} \neq 2A\frac{dA}{dt}$$

since above two matrices are not $=$!!

$\therefore$ in general,

$$\frac{d}{dt}A^n(t) \neq nA^{n-1}\frac{dA}{dt},$$

and so you can't differentiate $(*)$ term-by-term to get Ae^{At}.

$\boxed{4H-7}$

a) $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix} \ldots$$

similarly, $A^n = \begin{bmatrix} 1 & 0 \\ 2n & 1 \end{bmatrix}$

$$\therefore e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t & 0 \\ 2t & t \end{bmatrix} + \begin{bmatrix} t^2/2! & 0 \\ 4t^2/2! & t^2/2! \end{bmatrix} + \ldots$$

$$= \begin{bmatrix} 1+t+t^2/2!+\ldots & 0 \\ 2t+4t^2/2!+6t^3/3!+\ldots & 1+t+t^2/2!+\ldots \end{bmatrix}$$

But lower-left corner

$$= 2t\left(1+\frac{2t}{2!}+\frac{3t^2}{3!}+\frac{4t^3}{4!}+\ldots\right) = 2te^t$$

$$\therefore e^{At} = \begin{bmatrix} e^t & 0 \\ 2te^t & e^t \end{bmatrix} \quad (*)$$

b) $e^{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}t} = e^{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}t} \cdot e^{\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}t}$

$$= \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} \quad \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2t & 0 \end{bmatrix}\right)$$

(see book ex. 1 p. 451) (higher powers of mx. are 0)

$$= \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 2t & 1 \end{bmatrix} = (*)$$

c) Find F by solving the system:

$$\begin{aligned} x' &= x \\ y' &= 2x+y \end{aligned} \Rightarrow \begin{aligned} x &= c_1e^t \\ y'-y &= 2c_1e^t \end{aligned}$$

solving 2^{nd} equation as a linear eqn:

$$(ye^{-t})' = 2c_1$$

$$ye^{-t} = 2c_1t + c_2$$

$$y = c_1 \cdot 2te^t + c_2e^t$$

$$\therefore F = \begin{bmatrix} e^t & 0 \\ 2te^t & e^t \end{bmatrix} \quad F(0)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so $\quad e^{At} = F \cdot F(0)^{-1} = \begin{bmatrix} e^t & 0 \\ 2te^t & e^t \end{bmatrix}$

$\boxed{4I-1}$

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\vec{x} + \begin{bmatrix} -5 \\ -8 \end{bmatrix}t + \begin{bmatrix} 2 \\ -8 \end{bmatrix}$$

① Solve the reduced equation $\vec{x}' = A\vec{x}$

· char. eqn is $m^2+m-6=0$, $(m+3)(m-2)=0$ roots: $m=-3$, $m=2$

$\underline{m=-3}$: $4\alpha_1+\alpha_2=0$ soln: $\begin{bmatrix} 1 \\ -4 \end{bmatrix}e^{-3t}$ $\quad \underline{m=2}$: $-\alpha_1+\alpha_2=0$ soln: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t}$

$\underline{\text{Fund mx}}$: $\begin{bmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{bmatrix} = F$. $\quad F^{-1} = \begin{bmatrix} e^{2t} & -e^{2t} \\ 4e^{-3t} & e^{-3t} \end{bmatrix}\frac{1}{5e^{-t}}$

$$v' = F^{-1}\begin{bmatrix} -5t+2 \\ -8t-8 \end{bmatrix} = \begin{bmatrix} \frac{e^{3t}}{5}(-5t+2) - \frac{e^{3t}}{5}(-8t-8) \\ \frac{4}{5}e^{-2t}(-5t+2) + \frac{e^{-2t}}{5}(-8t-8) \end{bmatrix} = \begin{bmatrix} \frac{3e^{3t}}{5}t + 2e^{3t} \\ -\frac{28}{5}e^{-2t}t \end{bmatrix}$$

$$\therefore v = \begin{bmatrix} \frac{te^{3t}}{5} + \frac{3}{5}e^{3t} \\ \frac{14}{5}e^{-2t}t + \frac{7}{5}e^{-2t} \end{bmatrix}$$

$$\vec{x}_p = F\vec{v} = \begin{bmatrix} \frac{t}{5}+\frac{3}{5}+\frac{14}{5}t+\frac{7}{5} \\ -\frac{4t}{5}-\frac{12}{5}+\frac{14}{5}t+\frac{7}{5} \end{bmatrix} = \begin{bmatrix} 3t+2 \\ 2t-1 \end{bmatrix} \text{ Ans.}$$

$\boxed{4I-2}$

a) Using the work from above:

$$v' = \frac{1}{5}\begin{bmatrix} e^{3t} & -e^{3t} \\ 4e^{-2t} & e^{-2t} \end{bmatrix}\begin{bmatrix} e^{-2t} \\ -2e^{t} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} e^{t}+2e^{4t} \\ 4e^{-4t}-2e^{-t} \end{bmatrix}$$

$$v = \frac{1}{5}\begin{bmatrix} e^t + \frac{e^{4t}}{2} \\ -e^{-4t}+2e^{-t} \end{bmatrix} \qquad \vec{x} = Fv = \frac{1}{5}\begin{bmatrix} e^{-2t}+e^t/2 - e^{-2t}+2e^t \\ -4e^{-2t}-2e^t - e^{-2t}+2e^t \end{bmatrix}$$

$$\therefore \vec{x}_p = \frac{1}{5}\begin{bmatrix} 5/2\, e^t \\ -5e^{-2t} \end{bmatrix} = \begin{bmatrix} e^t/2 \\ -e^{-2t} \end{bmatrix}$$

Add to $\vec{x}_p$ the $\vec{x}_h = c_1\begin{bmatrix} 1 \\ -4 \end{bmatrix}e^{-3t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t}$

$\vec{x}_p = \vec{c}e^{-2t} + \vec{d}e^t$ Substitute in to the equation:

$$-2\vec{c}e^{-2t} + \vec{d}e^t = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\vec{c}e^{-2t} + \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\vec{d}e^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}e^{-2t} + \begin{bmatrix} 0 \\ -2 \end{bmatrix}e^t$$

$$\therefore -2\vec{c} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\vec{c} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and $\quad \vec{d} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\vec{d} + \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

Writing the left side of the 1^{st} system as $-2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\vec{c}$, it becomes (I'm just being cute — you could just write it all out + hack away) on subtracting $\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\vec{c}$ from both sides

$$\begin{bmatrix} -3 & -1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{or:} \quad \begin{aligned} -3c_1 - c_2 &= 1 \\ -4c_1 &= 0 \end{aligned} \quad \therefore \begin{aligned} c_1 &= 0 \\ c_2 &= -1 \end{aligned}$$

Similarly for the other system:

$$\begin{bmatrix} 0 & -1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \qquad \begin{aligned} -d_2 &= 0 \\ -4d_1 + 3d_2 &= -2 \end{aligned} \qquad \therefore \begin{aligned} d_2 &= 0 \\ d_1 &= 1/2 \end{aligned}$$

Thus $\quad \vec{x}_p = \begin{bmatrix} 0 \\ -1 \end{bmatrix}e^{-2t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}e^t = \begin{bmatrix} e^t/2 \\ -e^{-2t} \end{bmatrix}$ same as before, ta-da.

$\boxed{4I-4}$

Solve reduced equation first: $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{x}$

char eqn: $m^2 - 1 = 0$

$\underline{m=1}$: $\alpha_1 - \alpha_2 = 0$ $\qquad$ $\underline{m=-1}$: $3\alpha_1 - \alpha_2 = 0$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t$ soln. $\qquad$ $\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$ soln.

To find particular sol'n, since $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t$ is a soln of reduced equation, we have to use as the trial soln not just $\vec{c}e^t$ but

$$\vec{x}_p = \vec{c}e^t + \vec{d}te^t$$

Substituting into the ODE's:

$$\vec{c}e^t + \vec{d}e^t + \vec{d}te^t = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}(\vec{c}e^t + \vec{d}te^t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$$

$$\therefore \quad \vec{c} + \vec{d} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{c} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{d} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{d} \quad ..$$

Solving second system: (as done in prob (2b)) $\quad \begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = 0 \quad \vec{d} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} k$

solving first system: $\quad \begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1-k \\ -1-k \end{bmatrix}$

Subtract $3\times$ first row from second: $\quad \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1-k \\ -4+2k \end{bmatrix} \quad \therefore k = 2$

get: $-c_1 + c_2 = -1$ $\quad$ so take $\vec{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (other $\vec{c}$ are possible)

Soln: $\vec{x}_g = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \end{bmatrix} te^t + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$

to this could be added

$\boxed{4I-5}$

$\vec{x}' = A\vec{x} + \vec{X}_0$. Try $\vec{x}_p = \vec{c}$. Substituting:

$A\vec{c} + \vec{X}_0 = 0$. $\quad \therefore \quad \vec{x}_p = -A^{-1}\vec{X}_0$ if A is nonsingular!

[If A is singular, you only get soln $\vec{x}_p = \vec{c}$ if $A\vec{c} = -\vec{X}_0$ is consistent. In general, if rank $A = n - r$, you use $\vec{x}_p = \vec{c}_0 + \vec{c}_1 t + \dots + \vec{c}_{r-1} t^{r-1}$