Section 6 Solutions

6A-1 All of these use the ratio test:
if $\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = L$, $\sum_0^\infty b_n$ converges if $L<1$, diverges if $L>1$.

a) $$\left|\frac{(n+1)x^{n+1}}{n x^n}\right| = \left(\frac{n+1}{n}\right)|x| \longrightarrow |x| \quad \text{as } n\to\infty$$

$\therefore$ converges if $|x|<1$, so $R=1$

b) $$\left|\frac{x^{2(n+1)}}{(n+1)2^{n+1}}\cdot\frac{n\cdot 2^n}{x^{2n}}\right| = \frac{n}{(n+1)2}\cdot|x|^2$$

$$\underset{\text{as } n\to\infty}{\longrightarrow} \frac{1}{2}|x|^2, \text{ and } \frac{|x|^2}{2}<1 \text{ if } |x|<\sqrt{2}$$

$\therefore$ converges if $|x|<\sqrt{2}$, so $R=\sqrt{2}$

c) $$\left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| = (n+1)|x| \longrightarrow \infty \quad \text{as } n\to\infty$$
(if $x\neq 0$).

$\therefore$ converges only when $x=0$; $R=0$.

d) $$\left|\frac{[2(n+1)]!}{(n+1)!^2}\cdot x^{n+1}\cdot\frac{(n!)^2}{(2n)!\,x^n}\right|$$

$$= \frac{(2n+2)(2n+1)}{(n+1)(n+1)}\,|x| \longrightarrow 4|x| \quad \text{as } n\to\infty$$

$\therefore$ converges if $4|x|<1$, i.e., $|x|<\frac{1}{4}$, so $R = 1/4$

6A-2 a) $$\frac{1}{1-x} = \sum_0^\infty x^n$$

$$\therefore \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = \sum_1^\infty n x^{n-1} = \sum_0^\infty (n+1)x^n$$

(replacing n by $n+1$).

b) $$e^x = \sum_0^\infty \frac{x^n}{n!}, \quad \therefore e^{-x^2} = \sum_0^\infty \frac{(-1)^n x^{2n}}{n!}$$

$$x e^{-x^2} = \sum_0^\infty \frac{(-1)^n x^{2n+1}}{n!}$$

6A-2 c) $$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} = \sum_0^\infty (-1)^n x^{2n}$$

Integrating:
$$\tan^{-1}x = \sum_0^\infty \frac{(-1)^n x^{2n+1}}{2n+1} + \overset{0}{c}$$

($c=0$: substitute $x=0$ on both sides to see that $c=0$)

d) $$\frac{d}{dx}\ln(1+x) = \frac{1}{1+x} = \sum_0^\infty (-1)^n x^n$$

Integrating:
$$\ln(1+x) = \sum_0^\infty \frac{(-1)^n x^{n+1}}{n+1} + c^{=0}$$

(see that $c=0$ by substituting $x=0$ on both sides)
[Series could also be written $\sum_1^\infty (-1)^{n-1}\frac{x^n}{n}$ (putting n for $n+1$)]

6A-3 a) $$y = \sum_0^\infty \frac{x^{2n+1}}{(2n+1)!}$$

$$y' = \sum_0^\infty \frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_0^\infty \frac{x^{2n}}{(2n)!}$$

$$y'' = \sum_1^\infty \frac{2n\,x^{2n-1}}{(2n)!} = \sum_1^\infty \frac{x^{2n-1}}{(2n-1)!}$$

the 0 term disappears

$$= \sum_0^\infty \frac{x^{2n+1}}{(2n+1)!} \quad \text{(changing } n\to n+1)$$

This shows $y''=y$, or $y''-y=0$.

b) $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!}$$

$$\therefore \frac{e^x - e^{-x}}{2} = \frac{2x}{2} + \frac{2x^3}{2\,3!} + \frac{2}{2}\frac{x^5}{5!} + \dots$$

$$= \sum_0^\infty \frac{x^{2n+1}}{(2n+1)!}.$$

4a) $$\sum_0^\infty x^{3n+2} = x^2\sum_0^\infty x^{3n} = x^2\cdot\frac{1}{1-x^3}$$

(since $\sum_0^\infty x^{3n} = \sum_0^\infty (x^3)^n = \frac{1}{1-(x^3)}$).

6A-4b) Start with $\sum_0^\infty x^n = \frac{1}{1-x}$

Integrate both sides:

$$\sum_0^\infty \frac{x^{n+1}}{n+1} = -\ln(1-x) + c^{=0} \quad \left(\text{substitute } x=0\right)$$

$$\therefore \sum_0^\infty \frac{x^n}{n+1} = -\frac{\ln(1-x)}{x}.$$

4c) Start with $\sum_0^\infty x^n = \frac{1}{1-x}$

Differentiating, $\sum_1^\infty n x^{n-1} = \frac{1}{(1-x)^2}$

$$\therefore \sum_1^\infty n x^n = \frac{x}{(1-x)^2}$$

note $\to$ 1 or 0 (makes no difference)

6B-1

a) Since $y(0)=1$,

$$y = 1 + a_1 x + a_2 x^2 + a_3 x^3$$
$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3$$

$$y^2 = (1 + a_1 x + a_2 x^2 + \dots)(1 + a_1 x + a_2 x^2 + \dots)$$
$$= 1 + 2a_1 x + (2a_2 + a_1^2)x^2 + (2a_3 + 2a_2 a_1)x^3 + \dots$$

(this is far enough to get a_3.)

$y' = x + y^2$ says that

$$a_1 + 2a_2 x + 3a_3 x^2 + \dots = 1 + (2a_1 + 1)x + (2a_2 + a_1^2)x^2 + \dots$$

$\therefore$ equating coefficients of like powers of x gives us:

$$a_1 = 1, \quad 2a_2 = 2a_1 + 1 = 3, \quad \therefore a_2 = 3/2$$
$$3a_3 = 2a_2 + a_1^2 = 4, \quad \therefore a_3 = 4/3$$

So: $\boxed{y = 1 + x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \dots}$

b) Using Taylor's formula: $y(0)=1$

$y' = x + y^2 \qquad \therefore y'(0) = 0 + 1^2 = 1$

$\therefore y'' = 1 + y' \cdot 2y \qquad y''(0) = 1 + 1\cdot(2\cdot 1) = 3$

$y''' = y'' \cdot 2y + y' \cdot 2y' \qquad y'''(0) = 3\cdot 2 + 1 \cdot 2 = 8$

By Taylor's formula,

$$y = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \dots$$
$$\therefore y = 1 + x + \frac{3}{2}x^2 + \frac{8}{6}x^3 + \dots$$

just as in part (a).

6B-2

a) $y = \sum_0^\infty a_n x^n$

$y' = \sum_{0\,(\text{or }1)}^\infty n a_n x^{n-1} \to \sum_0^\infty (n+1)a_{n+1} x^n$

$y' - y = x$ says that

$$(n+1)a_{n+1} - a_n = 0 \quad \text{if } n \neq 1$$
$$= 1 \quad \text{if } n = 1,$$

that is, (since $y(0)=0$):

$$a_0 = 0, \quad a_{n+1} = \frac{a_n}{n+1} \quad \text{if } n \neq 1$$
$$\text{and} \quad 2a_2 - a_1 = 1.$$

This gives:

$$a_0 = 0, \ a_1 = 0, \ a_2 = 1/2, \ a_3 = \tfrac{1}{3}\cdot\tfrac{1}{2},$$
$$a_4 = \tfrac{1}{4}\cdot\tfrac{1}{3}\cdot\tfrac{1}{2}, \ \text{etc} \dots$$

so $y = \sum_{n=2}^\infty \frac{x^n}{n!} = e^x - 1 - x$

b) $y = \sum_0^\infty a_n x^n \xrightarrow{\cdot x} xy = \sum_0^\infty a_n x^{n+1}$

$y' = \sum_{1\,(\text{or }0)}^\infty n a_n x^{n-1} \xrightarrow{n \to n+2} \sum_{-1} (n+2)a_{n+2} x^{n+1}$ $\leftarrow$ note

$y' = -xy \Rightarrow$

$$(n+2)a_{n+2} = -a_n \qquad n = 0, 1, 2, \dots$$
$$a_1 = 0 \qquad (\text{corresponding to } n = -1)$$
$$a_0 = 1 \qquad (\text{since } y(0) = 1)$$
$$\therefore a_{n+2} = \frac{-a_n}{n+2} \qquad n = 0, 1, 2, \dots$$

so

$$a_0 = 1, \ a_2 = \frac{-1}{2}, \ a_4 = \frac{1}{4}\cdot\frac{1}{2}, \ a_6 = \frac{-1}{6\cdot 4\cdot 2}$$
$$a_1 = a_3 = a_5 = \dots = 0.$$

So $y = \sum_0^\infty \frac{x^{2n}(-1)^n}{2^n \cdot n!} = e^{-x^2/2}$

6B-2

c) $y = \sum_0^\infty a_n x^n$

$y' = \sum_{0\,(\text{or }1)}^\infty n a_n x^{n-1} \rightsquigarrow \sum_0^\infty (n+1) a_{n+1} x^n$

$x y' = \sum_0^\infty n a_n x^n$

$\therefore (1-x)y' - y = 0 \implies$ (equating the coeff of x^n to 0)

$(n+1)a_{n+1} - n a_n - a_n = 0$

or $a_{n+1} = \frac{(n+1)a_n}{n+1} = a_n$

$y(0) = 1 \implies a_0 = 1$

$\therefore y = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$

6C-1

a) $\sum_1^\infty a_n x^{n+3} \underset{n \to n-3}{=} \sum_4 a_{n-3} x^n$

this starts with x^4, so this must also

b) $\sum_0^\infty n(n-1) a_n x^{n-2} \underset{n \to n+2}{=} \sum_0^\infty (n+2)(n+1) a_{n+2} x^n$

starts with x^0, so this must also

c) $\sum_1^\infty (n+1) a_n x^{n-1} \underset{n \to n+1}{=} \sum_0^\infty (n+2) a_{n+1} x^n$

starts with x^0, so this must also

6C-2

$y = \sum_0^\infty a_n x^n \underset{\cdot 4}{\rightsquigarrow} 4y = \sum_0^\infty 4 a_n x^n$

$y'' = \sum_0^\infty a_n \cdot n(n-1) x^{n-2} \underset{n \to n+2}{\rightsquigarrow} \sum_0^\infty a_{n+2}(n+2)(n+1) x^n$

$y'' - 4y = 0 \implies a_{n+2}(n+2)(n+1) - 4a_n = 0$

or $\boxed{a_{n+2} = \frac{4\,a_n}{(n+2)(n+1)}}$ Recursion formula

$\therefore a_2 = \frac{4a_0}{2\cdot 1}, \quad a_4 = \frac{4}{4\cdot 3}\cdot\frac{4}{2\cdot 1} a_0 = \frac{4^2}{4!} a_0$

$a_3 = \frac{4a_1}{3\cdot 2} \quad a_5 = \frac{4}{5\cdot 4}\cdot\frac{4}{3\cdot 2}\cdot a_1 = \frac{4^2 a_1}{5!}$

continued above

6C-2

(continued)

get one series by taking $a_0 = 1, a_1 = 0$:

$y_0 = 1 + \frac{4}{2!}x^2 + \frac{4^2}{4!}x^4 + \frac{4^3}{6!}x^6 + \dots$

other series: take $a_0 = 0, a_1 = 1$

$y_1 = x + \frac{4x^3}{3!} + \frac{4^2 x^5}{5!} + \dots$

In summation notation:

$y_0 = \sum_0^\infty \frac{4^n x^{2n}}{n!}, \quad y_1 = \sum_0^\infty \frac{4^n x^{2n+1}}{(2n+1)!}$

(can also write numerator as $(2x)^{2n}$)

6C-3

Not solved.

6C-4

$\boxed{y'' - 2xy' + ky = 0}$; $\boxed{k = 2m}$.

$y = \sum_0^\infty a_n x^n \underset{\cdot 2m}{\rightsquigarrow} \sum_0^\infty 2m a_n x^n$

$y' = \sum_0^\infty n a_n x^{n-1} \underset{\cdot -2x}{\rightsquigarrow} \sum_0^\infty -2n a_n x^n$

$y'' = \sum_0^\infty n(n-1) a_n x^{n-2} \underset{n \to n+2}{\rightsquigarrow} \sum_0^\infty (n+2)(n+1) a_{n+2} x^n$

Since $y'' - 2xy' + ky = 0$, this gives

$(n+2)(n+1)a_{n+2} - 2n a_n + 2m a_n = 0$

or $\boxed{a_{n+2} = \frac{2(n-m)}{(n+2)(n+1)} a_n}$

If $n = m$, then $a_{m+2} = 0$, etc.

So: if m is odd,

take $a_0 = 0, a_1 = 1$; then

all $a_0 = a_2 = a_4 = \dots = 0$

and all $a_{m+2} = a_{m+4} = 0 = .$

so $y_1 = a_1 x + a_3 x^3 + \dots + a_m x^m$

If m is even, take $a_1 = 0$.

Then similarly, (so $a_3 = 0, a_5 = 0 ..$)

$y_0 = a_0 + a_1 x^2 + \dots + a_m x^m$

$\boxed{6C\text{-}5}$

$$\boxed{y'' = xy}$$

$$y = \sum_0^\infty a_n x^n \;\xrightarrow[\cdot x]{xy:}\; \sum_0^\infty a_n x^{n+1} = \sum_1^\infty a_{n-1} x^n$$

$$y'' = \sum_0^\infty n(n-1)a_n x^{n-2} \;\underset{n\to n+2}{\rightsquigarrow}\; \sum_0^\infty (n+2)(n+1)a_{n+2} x^n$$

Equating coeff's of like powers of x (since $y'' = xy$) gives

$$(n+2)(n+1)a_{n+2} = a_{n-1} \quad (n \geq 1) \quad \rightsquigarrow \therefore \boxed{a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}}, \quad n \geq 1.$$
$$= 0 \quad (n = 0) \qquad\qquad a_2 = 0$$

Recursion formula

$\therefore a_0, a_1$ are arbitrary, $2a_2 = 0$ (so $a_2 = 0$),

($\therefore a_5 = a_8 = a_{11} = \ldots = 0$ by the recursion formula)

and other terms are: $a_3 = \dfrac{a_0}{3\cdot 2}$, $a_6 = \dfrac{a_0}{6\cdot5\cdot3\cdot2} \ldots$

$a_4 = \dfrac{a_1}{4\cdot 3}$, $a_7 = \dfrac{a_1}{7\cdot6\cdot4\cdot3} \ldots$

Taking $a_0 = 1$, $a_1 = 0$ gives $y_0 = 1 + \dfrac{x^3}{3\cdot2} + \dfrac{x^6}{6\cdot5\cdot3\cdot2} + \cdots + \dfrac{x^{3n}}{3n\cdot(3n-1)(3n-3)\ldots3\cdot2} + \ldots$

taking $a_0 = 0$, $a_1 = 1$ gives $y_1 = x + \dfrac{x^4}{4\cdot3} + \dfrac{x^7}{7\cdot6\cdot4\cdot3} + \cdots + \dfrac{x^{3n+1}}{(3n+1)\cdot3n\cdot(3n-2)\cdots4\cdot3} + \ldots$

$\boxed{6C\text{-}6}$

$$\boxed{y'' - x^2y'' - 2xy' + 6y = 0}$$

$$y = \sum_0^\infty a_n x^n \rightsquigarrow 6y = \sum_0^\infty 6a_n x^n$$
$$y' = \sum_0^\infty n a_n x^{n-1} \rightsquigarrow -2xy' = \sum_0^\infty -2n a_n x^n$$
$$y'' = \sum_0^\infty n(n-1)a_n x^{n-2} \rightsquigarrow y'' = \sum_0^\infty (n+2)(n+1)a_{n+2}x^n$$
$$\rightsquigarrow -x^2y'' = -\sum_0^\infty n(n-1)a_n x^n$$

Equating coeff of x^n to 0 gives:

$$(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + 6a_n = 0$$

or $a_{n+2} = a_n \dfrac{[n(n-1) + 2n - 6]}{(n+2)(n+1)}$

or $\boxed{a_{n+2} = \dfrac{(n+3)(n-2)}{(n+2)(n+1)}a_n}$

RECURSION FORMULA.

This gives solutions

$y_0 = 1 - 3x^2 \quad (a_0 = 1,\ a_1 = 0 = a_3 = a_5 = \ldots)$

$y_1 = x - \frac{2}{3}x^3 - \frac{1}{5}x^5 - \frac{4}{35}x^7 - \cdots \quad \Leftarrow$

Radius of convergence for y_1 is determined by ratio test: $\left|\dfrac{a_{n+2}x^{n+2}}{a_n x^n}\right| = \dfrac{(n+3)(n-2)}{(n+2)(n+1)}|x|^2 \xrightarrow[\text{as } n\to\infty]{} x^2$, $|x|^2 < 1$ if $|x| < 1$

$\therefore R = 1$. This is expected, since in standard form, ODE is $y'' - \frac{2x}{1-x^2}y' + \frac{6}{1-x^2}y = 0$, and coefficients become infinite at $|x| = 1$.

$\boxed{6C\text{-}7}$

$$y = \sum_0^\infty a_n x^n \rightsquigarrow xy = \sum_1^\infty a_{n-1}x^n$$
$$y' = \sum_0^\infty n a_n x^{n-1} \rightsquigarrow 2y' = 2\sum_0^\infty (n+1)a_{n+1}x^n$$
$$y'' = \sum_0^\infty (n+2)(n+1)a_{n+2}x^n$$

$\therefore y'' + 2y' + (x-1)y = 0$ leads to the recursion:

$$(n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1} + a_{n-1} - a_n = 0$$

leading to two sol'ns: $y_0 = 1 + x^2/2 - x^3/2 + \ldots \quad \left(\begin{smallmatrix}a_0 = 1\\ a_1 = 0\end{smallmatrix}\right)$

$y_1 = x - x^2 + \frac{5}{6}x^3 + \ldots \quad (a_0 = 0, a_1 = 1)$