Recitation 10, March 9, 2010

Gain and phase lag; resonance; undetermined coefficients

Solution suggestions:

2. Explain why a spring/mass/dashpot system driven through the spring is modeled by the equation $m\ddot{x} + b\dot{x} + kx = ky$. Here x measures the position of the mass, y measures the position of the other end of the spring, and $x = y$ when the spring is relaxed.

As indicated, we set up the functions $x(t)$ and $y(t)$ so that when $x(t) = y(t)$ the spring is relaxed. Namely, let $w(t)$ denote the position of the other end point of the spring at moment t, and l the natural length of the spring when it's relaxed. Set $y(t) = w(t) + l$. When $x(t) = y(t)$, $x(t) - w(t) = l$, the spring is at its natural length, which implies it's relaxed. When $x(t) > y(t)$, $x(t) - w(t) > l$, the spring is streched and the spring force is negative; when $x(t) < y(t)$, $x(t) - w(t) < l$, the spring is compressed and the force is positive. So $m\ddot{x} + b\dot{x} = k(y - x)$, or $m\ddot{x} + b\dot{x} + kx = ky$.

3. In this system, regard $y(t)$ as the input signal and $x(t)$ as the system response. Take $m = 1$, $b = 3$, $k = 4$, $y(t) = A\cos t$. Replace the input signal by a complex exponential y_{cx} of which it is the real part, and use the Exponential Response Formula to compute the exponential ("steady state") system response z_p. Compute H such that $z_p = Hy_{cx}$; H is the *complex gain.* Find $|H|$ and ϕ such that $H = |H|e^{-i\phi}$. Use this information to compute the gain and the phase lag of the original system. What is the steady state solution? Is the amplitude of vibration of the mass greater than or less than the amplitude A of the motion of the far end of the spring?

The equation is $\ddot{x} + 3\dot{x} + 4x = 4A\cos t$, with the characteristic polynomial $p(s) = s^2 + 3s + 4$. The complex exponential corresponding to the input signal is $y_{cx} = Ae^{it}$ and $p(i) = 3 + 3i \neq 0$. By the Exponential Response Formula, $z_p = \frac{4A}{p(i)}e^{it} = \frac{4A}{3+3i}e^{it}$. So the complex gain is $H = z_p/y_{cx} = \frac{4}{3+3i} = \frac{2\sqrt{2}}{3}e^{-i\pi/4}$, and $|H| = 2\sqrt{2}/3$, $\phi = \pi/4$. The steady state solution of the original system is given by $x_p = Re(z_p) = \frac{2\sqrt{2}A}{3}\cos(t - \frac{\pi}{4})$. Therefore, the gain is $2\sqrt{2}/3$ and the phase lag is $\pi/4$. The amplitude of vibration of the mass is $2\sqrt{2}A/3$, which is less than A.

4. Find the polynomial solution of $\ddot{x} - x = t^2 + t + 1$.

Since the constant term of $p(D)$ is nonzero, the undetermined coefficients theorem asserts that there is a unique quadratic polynomial $at^2 + bt + c$ satisfying this equation. Substituting this form into the left side of the equation, we see that $a = -1$, $-b = 1$, and $2a - c = 1$, so $b = -1$ and $c = -3$.

5. Find a solution of $\ddot{x} + 4x = \cos(2t)$ by attempting to use the ERF on a complex replacement.

The complex replacement of the equation is $\ddot{z} + 4z = e^{2it}$, with the characteristic polynomial $p(s) = s^2 + 4$. Because $p(2i) = 0$ and $p'(2i) = 4i \neq 0$, we need to use the Resonant ERF, which leads to $z_p = \frac{te^{2it}}{4i}$. A solution of the original equation is given by $x_p = Re(z_p) = \frac{t}{4}\sin(2t)$.