18.085 Computational Science and Engineering I
Fall 2008

Gradient and Divergence / Parallel Table

Gradient	**Divergence**
$v = \operatorname{grad} u = \nabla u$	$\operatorname{div} w = \nabla \cdot w = 0$
Potential $u(x,y)$: $v_1 = \dfrac{\partial u}{\partial x}, v_2 = \dfrac{\partial u}{\partial y}$	Stream function $s(x,y)$: $w_1 = \dfrac{\partial s}{\partial y}, w_2 = -\dfrac{\partial s}{\partial x}$
Test on v: $\dfrac{\partial v_2}{\partial x} - \dfrac{\partial v_1}{\partial y} = 0$	Test on w: $\dfrac{\partial w_1}{\partial x} + \dfrac{\partial w_2}{\partial y} = 0$
Irrotational: zero vorticity	Solenoidal: zero source
Zero circulation around loops:	Zero flux through loops:
$\int v \cdot t\, ds = \int v_1\, dx + v_2\, dy = 0$	$\int w \cdot n\, ds = \int w_1\, dy - w_2\, dx = 0$
Kirchhoff's Voltage Law	Kirchhoff's Current Law
Equipotentials $u(x,y) = \text{constant}$	Streamlines $s(x,y) = \text{constant}$
v is perpendicular to equipotentials	w is tangent to streamlines

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Green-Gauss Formula $\displaystyle\iint w \cdot \operatorname{grad} u\, dx\, dy = \iint u(-\operatorname{div} w)\, dx\, dy + \int u\, w \cdot n\, ds$

$(\textbf{grad})^{\mathrm{T}} = -\,\textbf{div}$ from integration by parts: $(\boldsymbol{Au})^{\mathrm{T}}\boldsymbol{w} = \boldsymbol{u}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{w})$

Connections when $(v_1, v_2) = (w_1, w_2)$

1. Equipotentials are perpendicular to streamlines
2. Laplace's equation $\operatorname{div}(\operatorname{grad} u) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right) = \nabla \cdot \nabla u = 0$
3. Cauchy-Riemann equations $\dfrac{\partial u}{\partial x} = \dfrac{\partial s}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial s}{\partial x}$ connecting u to s
4. Laplace's equation for s $\dfrac{\partial^2 s}{\partial x^2} + \dfrac{\partial^2 s}{\partial y^2} = -\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y \partial x} = 0$
5. Zero vorticity and zero source: Ideal potential flow
6. In two dimensions: $u(x,y) + is(x,y)$ is a function $f(x+iy)$