12/1.

Thue's Theorem.

$$\#\{\underset{\text{nonzero}}{a,b,c \in \mathbb{Z}} \mid ax^3 + by^3 = C\} \text{ is finite.}$$

$$\Updownarrow$$

$$x^3 - by^3 = c \qquad b, c > 0.$$

$$x^3 - by^3 = (x - \beta y)\,\underbrace{(x^2 + \beta xy + \beta^2 y^2)}_{\geq \frac{3}{4}\beta^2 y^2} \qquad \beta = \sqrt[3]{b}.$$

$$\left|\frac{x}{y} - \beta\right| \leq \frac{4|c|}{3\beta^2}\,\frac{1}{|y|^3}$$

DAT:

$\beta \equiv \sqrt[3]{b} \notin \mathbb{Z}$.

$$\#\{p, q \in \mathbb{Z} \mid q > 0,\ \left|\frac{p}{q} - \beta\right| \leq \frac{C}{q^3}\} < \infty \qquad \forall C.$$

(1) $F(x, y) \in \mathbb{Z}[x, y]$ that vanishes to high order at (β, β).

(2) Upper Bound for $\left|F\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right|$ in terms of $\left|\frac{p_1}{q_1} - \beta\right|$ and $\left|\frac{p_2}{q_2} - \beta\right|$.

(3) Lower bound for $\left|F\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)\right|$

Recall **APT**

$\beta \equiv \sqrt[3]{b} \quad m, n \in \mathbb{Z} \mid m + 1 > \frac{2}{3}n \geq m \geq 3.$

$$\exists F(x, y) \in \mathbb{Z}[x, y] \mid F(x, y) = P(x) + YQ(x)$$

$$= \sum_{i=0}^{m+n} u_i x^i + v_i x^i y.$$

$$F^{(k)}(\beta, \beta) = 0 \qquad 0 \leq k \leq n \qquad \max_{0 \leq i \leq m+n} \{|u_i|, |v_i| \leq 2(|b|)^{q(m+n)}\}$$

Recall

SWT:

$\exists\ c_1(b) > 0$ s.t. $\left|F^{(t)}(x,y)\right| \le c_1^n \{ |x-\beta|^{n-t} + |y-\beta| \}$

for all real x, y s.t. $|x-\beta| \le 1$, $\forall\ t \le n$.

Non Vanishing Theorem NVT. (Today).

$\frac{p_1}{q_1}, \frac{p_2}{q_2} \in \mathbb{Q}$ in lowest terms.

$\exists\ c_2(b) > 0$ and $t \in \mathbb{Z}$ s.t. $0 \le t \le 1 + \frac{c_2 n}{\log q_1}$,

$F^{(t)}\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) \ne 0.$

$T \equiv$ largest integer s.t. $F^{(t)}\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) =$

$$P^{(t)}\left(\frac{p_1}{q_1}\right) + Q^{(t)}\left(\frac{p_1}{q_1}\right)\left(\frac{p_2}{q_2}\right) = 0 \qquad 0 \le t < T$$

eliminate $\frac{p_2}{q_2}$ from each pair of equations. s, t

$$P^{(t)}\left(\frac{p_1}{q_1}\right) Q^{(s)}\left(\frac{p_1}{q_1}\right) - P^{(s)}\left(\frac{p_1}{q_1}\right) Q^{(t)}\left(\frac{p_1}{q_1}\right) = 0$$

$$0 \le s, t < T.$$

$$W_{P,Q}(x) = \begin{vmatrix} P(x) & Q(x) \\ P'(x) & Q'(x) \end{vmatrix} = P(x)\,Q'(x) - P'(x)\,Q(x).$$

$$W^{(r)}(x) = \sum_{i+j=r} \frac{i!\,(j+1)!}{r!} \left(P^{(i)}(x)\,Q^{(j+1)}(x) - Q^{(i)}(x)\,P^{(j+1)}(x) \right)$$

$x = \frac{p_1}{q_1}$ $\quad r < T-1 \quad W^{(r)}\left(\frac{p_1}{q_1}\right) = 0 \quad \forall\ 0 \le r \le T-1$

$$\left(x - \frac{p_1}{q_1}\right)^{T-1} \text{ is a factor of } W(x).$$

Gauss' Thm.

$P \in \mathbb{Z}[x]$ factors in $\mathbb{Q}[x]$

$\Rightarrow$ factors in $\mathbb{Z}[x]$.

$$W(x) = (q_1 x - p_1)^{T-1} V(x) \qquad V(x) \in \mathbb{Z}[x].$$

$$P(x) = \sum u_i x^i \qquad Q(x) = \sum v_i x^i$$

$$W(x) = \sum_{i,j} j\,(u_i v_j - v_i u_j)\, x^{i+j-1}$$

$$\max_{i,j \le n+m} |j\,(u_i v_j - v_i u_j)| \le 2(m+n)\left(\max_{i \le m+n} \{u_j, v_i\}\right)^2$$

$$\le 2(m+n)\left(2 \cdot (16b)^{9(m+n)}\right)^2 \qquad m \le \tfrac{2}{3} n.$$

$$\le C_3^{\,n} \qquad C_3 \text{ is a function of } b.$$

$q_1^{T-1} \le$ largest coefficient of Wronskian. $\le C_3^{\,n}$.

$$T \le 1 + \frac{c_2 n}{\log q_1} \qquad c_2 = \log C_3.$$

$$\exists\, t \mid 0 \le t \le 1 + \frac{c_2 n}{\log q_1} \qquad F^{(x)}\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) \ne 0.$$

So done <u>except</u> what if $W(x) = 0$?

Assume that $W(x) = 0$.

$$\Rightarrow P(x)\,Q'(x) = Q(x)\,P'(x)$$

$$\Rightarrow \frac{d}{dx}\left(\frac{P(x)}{Q(x)}\right) = 0.$$

$$P(x) = a\,Q(x). \qquad a \in \mathbb{Q}.$$

Then $F(x,y) = (a+y)\,Q(x)$

$$F^{(k)}(\beta,\beta) = \underbrace{(a+\beta)}_{\neq 0.}\,Q^{(k)}(\beta) = 0. \qquad 0 \le k < n.$$

$$\Downarrow$$

$$= 0.$$

$$(x-\beta)^n \mid Q(x).$$

$(x^3-b)^n \mid Q(x)$ $\qquad \deg Q(x) \ge 3n.$

but it isn't : $\deg Q \le m+n \le \frac{5}{3}n.$

So $W(x) \neq 0.$