TRANSFER FUNCTIONS

A differential equation of the form

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y$$
$$= b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1 \frac{du}{dt} + b_0 u$$

can be written more compactly as

$$\sum_{k=0}^{n} a_k \frac{d^k y}{dt^k} = \sum_{k=0}^{m} b_k \frac{d^k u}{dt^k}$$

The transfer function associated with this system may be found by taking the Laplace transform of both sides under the assumption of initial rest. In this case, a kth-order derivative is replaced by s^k to yield

$$a_n s^n Y(s) + a_{n-1} s^{n-1} Y(s) + \cdots + a_1 s Y(s)$$
$$= b_m s^m U(s) + b_{m-1} s^{m-1} U(s) + \cdots + b_0 U(s)$$

or

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0) Y(s)$$
$$= (b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0) U(s)$$

and thus

$$\sum_{k=0}^{n} a_k s^k Y(s) = \sum_{k=0}^{m} b_k s^k U(s)$$

We can define the two polynomials

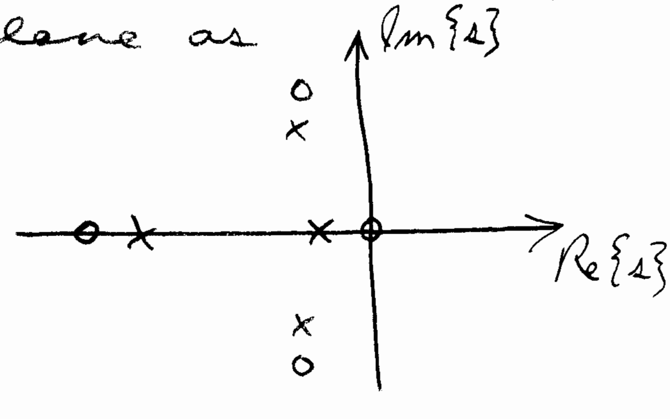
$$a(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

$$b(s) = b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0$$

Note that $a(s) = 0$ is the characteristic equation for the differential equation, and the roots of $a(s) = 0$ are the poles.

The roots of $b(s) = 0$ are the system zeros. The poles are the natural frequencies, and thus express the possible frequency content of the homogeneous response. The homogeneous response can be thought of as what you can have at the output of a system when the input is identically zero.

The zeros express the manner in which a system is connected to the outside world via inputs and outputs. The roots of $b(s) = 0$ are frequencies (in general complex-valued) for which you can have an input with no corresponding output. We plot the zeros as O's on the complex plane as



The poles are X's, as before.

The input/output relationship can be represented by the transfer function $H(s)$ which is the ratio $Y(s)/U(s)$

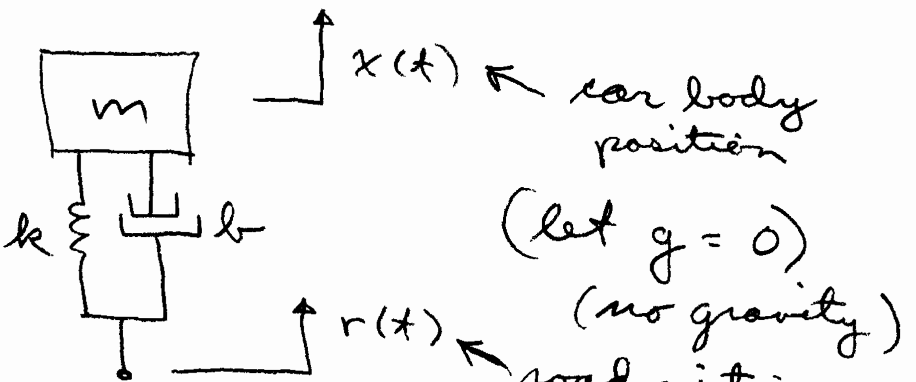
$$\frac{Y(s)}{U(s)} = \frac{b(s)}{a(s)} = \frac{b_m s^m + \cdots b_1 s + b_0}{a_n s^n + \cdots a_1 s + a_0} \triangleq H(s)$$

Another way of looking at this is

$$Y(s) = H(s)\,U(s)$$

That is, if we know $U(s)$ (the transform of the input), and $H(s)$ (the transform representation of the differential equation), then we can calculate $Y(s)$, the transform of the output. Then $Y(s)$ can be inverted to yield $y(t)$, the output as a function of time.

As an example, consider the simplified model of an automobile suspension



This has a differential equation

$$m\ddot{x} + b\dot{x} + kx = b\dot{r} + kr$$

Taking the transform of both sides under the assumption of initial rest yields

$$(ms^2 + bs + k)\,X(s) = (bs + k)\,R(s)$$

and thus

$$\frac{X(s)}{R(s)} = \frac{\text{Output}(s)}{\text{Input}(s)} = \frac{bs+k}{ms^2+bs+k} \triangleq H(s)$$

The poles are solutions of

$$a(s) = ms^2 + bs + k = 0 \quad \text{(roots of denominator)}$$

and thus are

$$s_1 = -\sigma + j\omega_d \quad [\text{sec}^{-1}]$$

$$s_2 = -\sigma - j\omega_d \quad [\text{sec}^{-1}]$$

where

$$\sigma = \zeta\omega_n = b/2m$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \frac{\sqrt{4mk - b^2}}{2m}$$

and

$$\zeta = \frac{b}{2\sqrt{km}}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

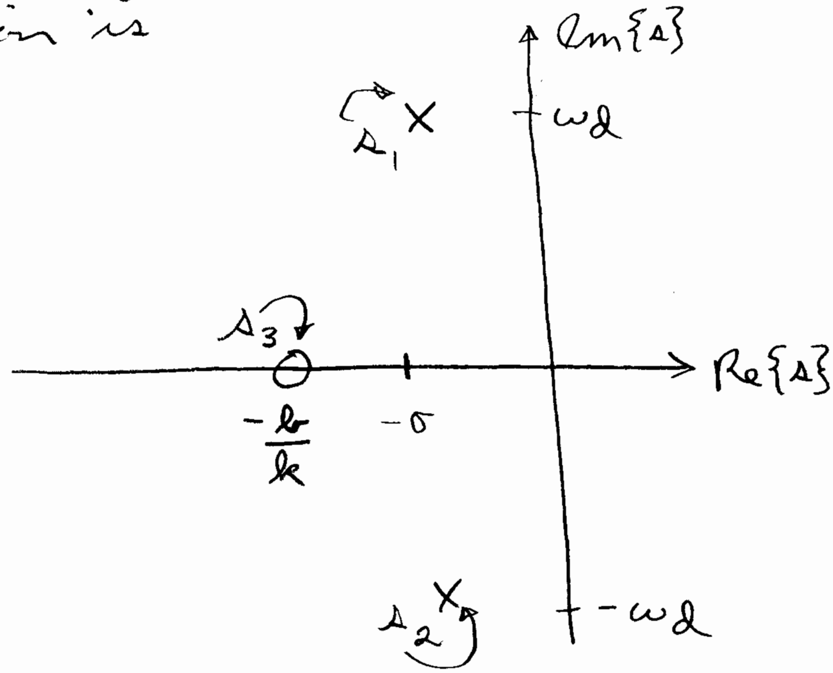
The zeros are solutions of

$$b(s) = bs + k = 0 \quad \text{(roots of numerator)}$$

and thus the one solution is

$$s_3 = -b/k \quad [\text{sec}^{-1}]$$

The pole-zero diagram for this transfer function is



Suppose we want to calculate the step response of this system from rest initial conditions. Then $R(s) = \mathcal{L}\{u_s(t)\} = 1/s$, and thus

$$X(s) = \frac{1}{s} \cdot \frac{bs + k}{ms^2 + bs + k}$$

There is no entry in our table for this transform (more extensive tables certainly exist), but we can get around this by expanding in simpler terms

$$X(s) = \frac{bs}{s(ms^2 + bs + k)} + \frac{k}{s(ms^2 + bs + k)}$$

$$= \frac{b}{ms^2 + bs + k} + \frac{k}{s(ms^2 + bs + k)}$$

$$\triangleq X_1(s) + X_2(s)$$

From the table

$$e^{-at} \sin bt \overset{\mathcal{L}}{\Longleftrightarrow} \frac{b}{(s+a)^2 + b^2} \qquad (1)$$

$$1 - e^{-at}\left(\cos bt + \frac{a}{b} \sin bt\right) \overset{\mathcal{L}}{\Longleftrightarrow} \frac{a^2+b^2}{s\left[(s+a)^2+b^2\right]} \qquad (2)$$

It takes a bit of algebraic juggling to get $X(s)$ into these forms. First $X_1(s)$:

$$X_1(s) = \frac{b}{ms^2+bs+k} = \frac{b/m}{\left(s+\frac{b}{2m}\right)^2 + \frac{4mk-b^2}{4m^2}}$$

$$= \frac{2\sigma}{(s+\sigma)^2 + \omega_d^2}$$ where σ, ω_d as defined earlier

$$= \frac{2\sigma}{\omega_d} \frac{\omega_d}{(s+\sigma)^2+\omega_d^2}$$

Using (1) above, we can invert to yield

$$x_1(t) = \frac{2\sigma}{\omega_d} e^{-\sigma t} \sin \omega_d t$$

Now, work on $X_2(s)$:

$$X_2(s) = \frac{k}{s\left[ms^2+bs+k\right]} = \frac{k/m}{s\left[(s+\sigma)^2+\omega_d^2\right]}$$

$$= \frac{\omega_n^2}{\sigma^2+\omega_d^2} \cdot \frac{\sigma^2+\omega_d^2}{s\left[(s+\sigma)^2+\omega_d^2\right]}$$

and thus, since $\frac{\omega_n^2}{\sigma^2+\omega_d^2} = 1$, we have via (2):

$$x_2(t) = 1 - e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right)$$

Combining gives

$$x(t) = x_1(t) + x_2(t)$$

$$= \frac{2\sigma}{\omega_d} e^{-\sigma t}\sin\omega_d t + 1 - e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right)$$

and thus

$$x(t) = 1 - e^{-\sigma t}\left(\cos\omega_d t - \frac{\sigma}{\omega_d}\sin\omega_d t\right)$$

Note: $x(0^+) = 0$

$$\dot{x}(0^+) = 2\sigma = \frac{b}{m}$$

and thus the velocity is discontinuous by an amount b/m across $t = 0$ (from 0^- to 0^+). This discontinuity is due to the force impulse from the damper $b\,\delta(t)$ acting on the mass.

This response is shown in Fig. 4 of the Laplace article.