2.003J/1.053J Dynamics and Control I, Spring 2007
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Lecture 4

Systems of Particles: Angular Momentum and Work Energy Principle

Systems of Particles

Angular Momentum (continued)

$$\boxed{\underline{\tau}_B^{ext} = \frac{d}{dt}\underline{H}_B + \underline{v}_B \times \underline{P}}$$

$\underline{\tau}_B^{ext}$: Total External Torque
$\underline{H}_B$: Total Angular Momentum
$\underline{P}$: Total Linear Momentum

From now on, $\underline{\tau}_B^{ext} = \underline{\tau}_B$.

If $\underline{\tau}_B = 0$ and $\underline{v}_B = 0$ or if B is the center of mass or if $\underline{v}_B \parallel \underline{v}_C$ then $\underline{H}_B =$ constant (Conservation of Angular Momentum).

You may be familiar with $\underline{\tau}_B = \frac{d}{dt}\underline{H}_B$ (only valid if $\underline{v}_B = 0$ or $\underline{v}_B \parallel \underline{P}$).
Angular momentum $\underline{H}_B$ of a collection of particles about point B is given by:

$$\underline{H}_B = \sum_{i=1}^{N} \underline{h}_{B_i}$$

where $\underline{h}_{B_i} = \underline{r}_i' \times m_i \underline{v}_i$.

If ($\underline{H}_B$) is the sum of the angular momenta of the individual particles about point B,

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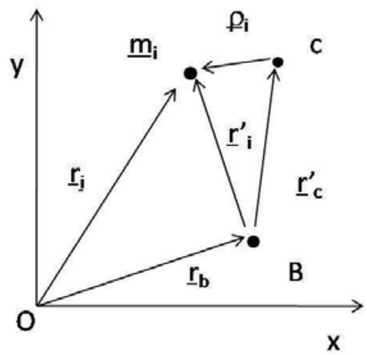

Figure 1: Angular momentum about B for a system of particles. Each particle has mass m_i positions r_i with respect to the origin and $r_i^{'}$ with respect to B. The center of mass C has positions $r_c^{'}$ with respect to B and ρ_i with respect to each point mass m_i. Figure by MIT OCW.

$$
\begin{aligned}
\underline{H}_B &= \sum_{i=1}^{n} \underline{r}_i^{'} \times m_i \underline{v}_i = \sum_{i=1}^{n} \underline{r}_i^{'} \times m_i \underline{\dot{r}}_i^{'} \\
&= \sum_{i=1}^{n} (\underline{r}_c^{'} + \underline{\rho}_i) \times m_i \underline{v_i} \\
&= \sum_{i=1}^{n} (\underline{r}_c^{'} \times m_i \underline{v}_i) + \sum_{i=1}^{n} \underline{\rho}_i \times m_i \underline{v}_i \\
&= \underline{r}_c^{'} \times M \underline{v}_c + \sum_{i=1}^{n} \underline{\rho}_i \times m_i \underline{v}_i
\end{aligned}
$$

where we have used $\sum m_i \underline{v}_i = M \underline{v}_c$

Therefore, we write:

$$
\boxed{\underline{H}_B = \underline{H}_C + \underline{r}_c^{'} \times \underline{P}}
$$

Notice that $\underline{v}_B$ does not appear in this equation.

The angular momentum about B is the angular momentum about the center of mass (C) plus the moment of the system linear momentum ($M\underline{v}_C = \underline{P}$) about B.

We will use these equations for rigid bodies. With rigid bodies will need to use moments of inertia.

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Work Energy Principle

$$W_{12} = \sum_{i=1}^{n} \int_{\underline{r}_{1i}}^{\underline{r}_{2i}} \underline{F}_i \cdot d\underline{r}_i = \sum_{i=1}^{n} \int_{t_1}^{t_2} \frac{d}{dt}\underline{p}_i \cdot \underline{v}_i dt = \int_{t_1}^{t_2} \sum_{i=1}^{n} \frac{d}{dt}\left[\frac{1}{2}m_i(\underline{v}_i \cdot \underline{v}_i)\right] dt$$

$$= T_2 - T_1$$

where:

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i (\underline{v}_i \cdot \underline{v}_i)$$

$$\begin{aligned} W_{12} &= \sum_{i=1}^{n} (W_{12})_i = \sum_{i=1}^{n} \int_{\underline{r}_{1i}}^{\underline{r}_{2i}} F_i \cdot d\underline{r}_i \\ &= \sum_{i=1}^{n} \int_{\underline{r}_{1i}}^{\underline{r}_{2i}} \underline{F}_i^{int} \cdot d\underline{r}_i + \sum_{i=1}^{n} \int_{\underline{r}_{1i}}^{\underline{r}_{2i}} \underline{F}_i^{ext} \cdot d\underline{r}_i \end{aligned}$$

$\sum_{i=1}^{n} \int_{\underline{r}_{1i}}^{\underline{r}_{2i}} \underline{F}_i^{int} \cdot d\underline{r}_i = W_{12}^{int}$
$\sum_{i=1}^{n} \int_{\underline{r}_{1i}}^{\underline{r}_{2i}} \underline{F}_i^{ext} \cdot d\underline{r}_i = W_{12}^{ext}$

$$\begin{aligned} W_{12}^{int} &= \sum_{i=1}^{n} \int_{t_1}^{t_2} \underline{F}_i^{int} \cdot \underline{v}_i dt \\ &= \sum_{i=1}^{n} \int_{t_1}^{t_2} \sum_{\substack{j=1 \\ j \neq 1}}^{n} f_{ij} \cdot \underline{v}_i dt \\ &= \int_{t_1}^{t_2} \sum_{i=1}^{n} \sum_{j>1}^{n} (\underline{f}_{ij} \cdot \underline{v}_i + \underline{f}_{ji} \cdot \underline{v}_j) dt \end{aligned}$$

$$W_{12}^{int} = \int_{t_1}^{t_2} \sum_{i=1}^{n} \sum_{j>1} \underline{f}_{ij} \cdot (\underline{v}_i - \underline{v}_j) dt$$

This is non-zero in general.

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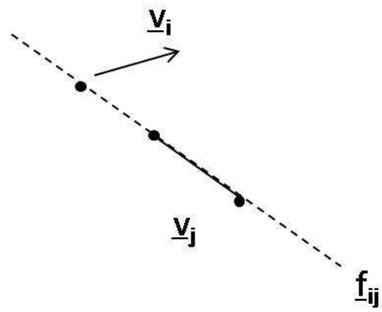

Figure 2: Relative velocity probably has a component in the direction of $\underline{f}_{ij}$. The figure shows two random points with randomly chosen velocities. Unless the difference between the velocities of the two points is zero or perpendicular to the direction of force $\underline{f}_{ij}$, $\underline{f}_{ij} \cdot (\underline{v}_i - \underline{v}_j)$ will not be zero; there would be some component in the direction of $\underline{f}_{ij}$. Figure by MIT OCW.

No reason that difference between velocities should not have a component in the direction of $\underline{f}_{ij}$.
If particles are parts of a rigid body system, then there is no relative motion in the direction of $\underline{f}_{ij}$ (e.g.)

Figure 3: Two point masses connected by a rod. This is an example of a rigid body where due to the rod, there is no relative motion of the two point masses at each end when the rigid body moves. Figure by MIT OCW.

$$\frac{d}{dt}|\underline{r}_i - \underline{r}_j|^2 = 0$$

$$\frac{d}{dt}\left[(\underline{r}_i - \underline{r}_j) \cdot (\underline{r}_i - \underline{r}_j)\right] = 0$$

$$2(\underline{r}_i - \underline{r}_j) \cdot (\underline{v}_i - \underline{v}_j) = 0$$

Internal forces $\underline{f}_{ij}$ are along the direction $(\underline{r}_i - \underline{r}_j)$.

$\underline{f}_{ij} \cdot (\underline{v}_i - \underline{v}_j) = 0.$

Therefore, for a rigid body system we have proved:

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$$\underline{f}_{ij} \cdot (\underline{v}_i - \underline{v}_j) = 0$$

Therefore, W_{12}^{int} must be 0 (or if you show that internal forces do no work).

Thus,

$$\boxed{W_{12}^{ext} = T_2 - T_1}$$

More generally:

$$W_{12}^{ext} + W_{12}^{int} = T_2 - T_1$$

If all external forces are potential forces or the ones who are external do no work and $W_{12}^{int} = 0$,

$$W_{12} = W_{12}^{ext} = V_1^{ext} - V_2^{ext}$$

$V =$ potential work where $V^{ext} = \sum_{i=1}^{n} V_i^{ext}$. V_i^{ext} is the external force potential of particle i.

$$\boxed{T_1 + V_1^{ext} = T_2 + V_2^{ext}}$$

Examples

Example 1

How does l affect the motion? How does θ affect the motion?

No rotations involved. Probably will not need angular momentum.

Kinematics

Describe the motion (kinematics) without forces
Knowing the location of A is equivalent to knowing the location of the center of mass of M.

$$\underline{v}_C = \underline{v}_A$$

M	m
$\underline{r}_A = x\hat{\imath}$	$\underline{r}_B = x\hat{\imath} + s\hat{e}_s = x\hat{\imath} + s\cos\theta\hat{\imath} + s\sin\theta\hat{\jmath}$
$\underline{\dot{r}}_A = \dot{x}\hat{\imath}$	$\underline{\dot{r}}_B = \dot{x}\hat{\imath} + \dot{s}\cos\theta\hat{\imath} + \dot{s}\sin\theta\hat{\jmath}$
$\ddot{r}_A = \ddot{x}\hat{\imath}$	$\underline{\ddot{r}_B} = \ddot{x}\hat{\imath} + \ddot{s}\cos\theta\hat{\imath} + \ddot{s}\sin\theta\hat{\jmath}$

Note: Generalized coordinates. $\hat{\imath}$ and $\hat{e}_s$ are not $\perp$.
Important to define coordinates.

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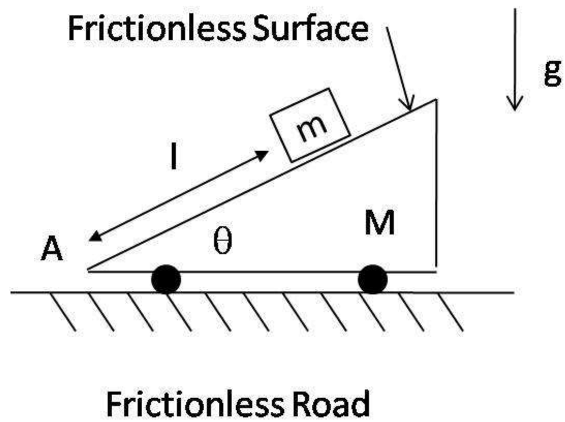

Figure 4: Block on frictionless surface that moves on frictionless road. The mass (m) can slide down the incline in a frictionless manner. Mass (M) is free to move horizontally without friction. If mass (m) is released from rest at the l position, find the velocity of mass (M) at the moment (m) reaches the bottom of the incline. Figure by MIT OCW.

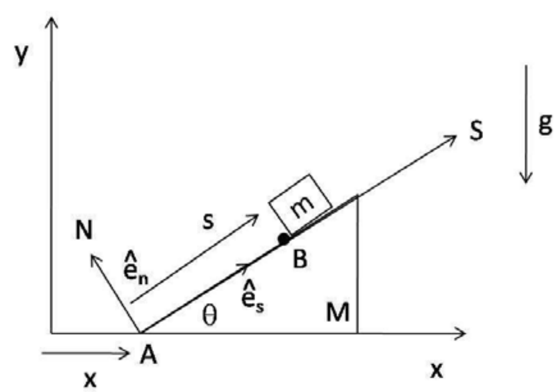

Figure 5: Diagram of kinematics of block on ramp. Need two sets of coordinates. M only moves in the x-direction. m only moves in the $\hat{e}_s$ direction. Figure by MIT OCW.

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