2.094— Finite Element Analysis of Solids and Fluids

— Fall ‘08 —

MIT OpenCourseWare

Contents

2.094 — Finite Element Analysis of Solids and Fluids Fall '08

Lecture 1 - Large displacement analysis of solids/structures

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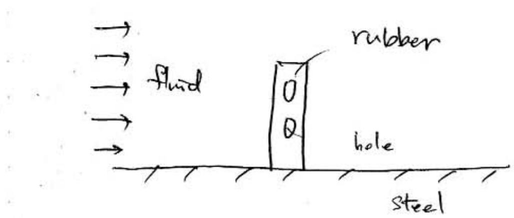
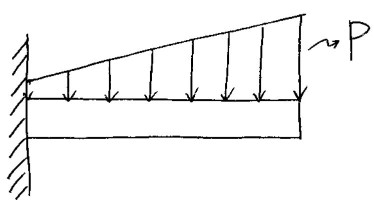
1.1 Project Example

Physical problem

Reading: Ch. 1 in the text

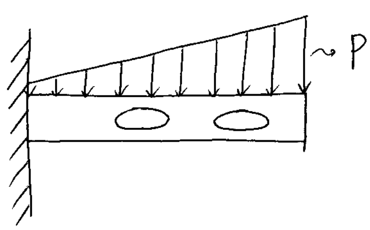


"Simple" mathematical model



- analytical solution
- F.E. solution(s)

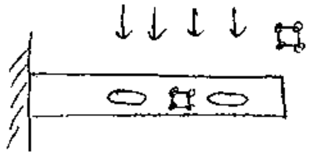
More complex mathematical model



- holes included
- large disp./large strains
- F.E. solution(s) $\Rightarrow$
- How many finite elements?

We need a good error measure (especially for FSI)

"Even more complex" mathematical model



The "complex mathematical model" includes Fluid Structure Interaction (FSI).

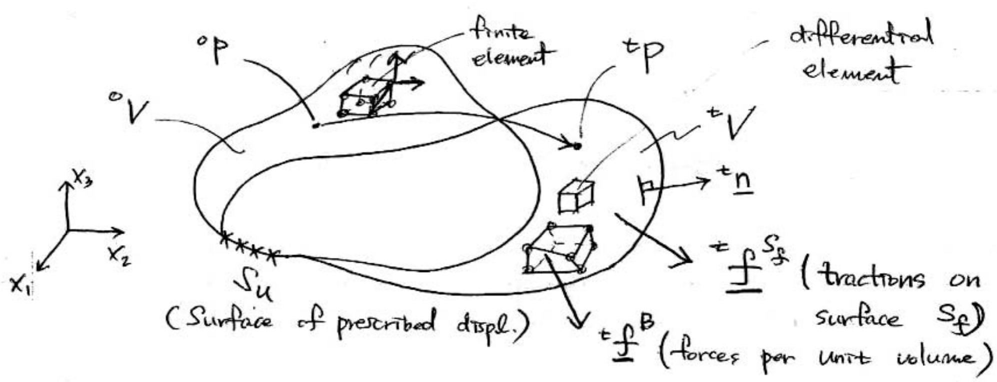
You will use ADINA in your projects (and homework) for structures and fluid flow.

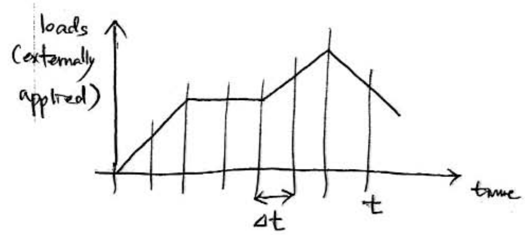
1.2 Large Displacement analysis

Lagrangian formulations:

- Total Lagrangian formulation
- Updated Lagrangian formulation

Reading: Ch. 6





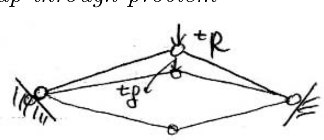
1.2.1 Mathematical model/problem

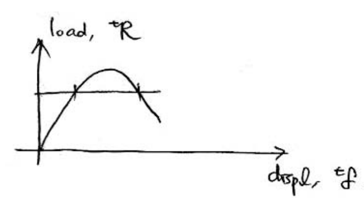
Given	the original configuration of the body,
	the support conditions,
	the applied external loads,
	the assumed stress-strain law
Calculate	the deformations, strains, stresses of the body.

Question Is there a unique solution? Yes, for infinitesimal small displacement/strain. Not necessarily for large displacement/strain.

For example:

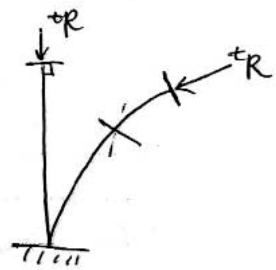
Snap-through problem



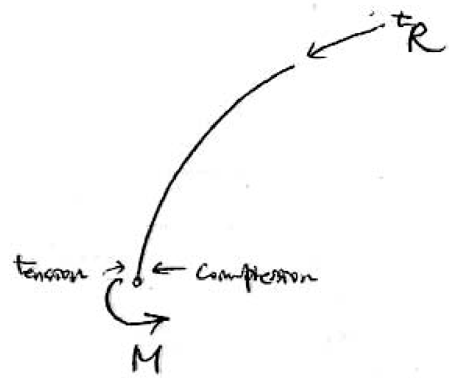


The same load. Two different deformed configurations.

Column problem, statics



Not physical



${}^tR$ is in "direction" of bending moment $\Rightarrow$ Not in equilibrium.

1.2.2 Requirements to be fulfilled by solution at time t

I. Equilibrium of stresses (Cauchy stresses, forces per unit area in ${}^tV$ and on ${}^tS_f$) with the applied body forces ${}^t\boldsymbol{f}^B$ and surface tractions ${}^t\boldsymbol{f}^{S_f}$

II. Compatibility

III. Stress-strain law

1.2.3 Finite Element Method

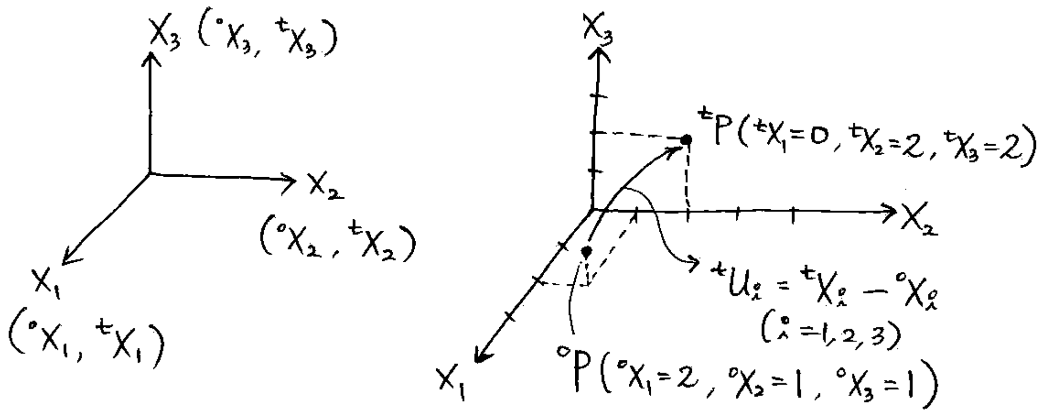
I. Equilibrium condition means now

- equilibrium at the nodes of the mesh
- equilibrium of each finite element

II. Compatibility satisfied exactly

III. Stress-strain law satisfied exactly

1.2.4 Notation



Cauchy stresses (force per unit area at time t):

$$ {}^{t}\tau_{ij} \quad i,j=1,2,3 \quad {}^{t}\tau_{ij} = {}^{t}\tau_{ji} \tag{1.1}$$

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Lecture 2 - Finite element formulation of solids and structures

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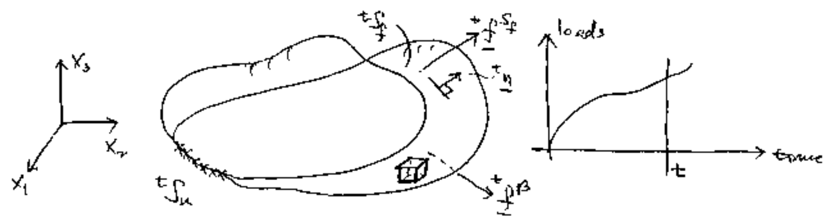


Reading: Ch. 1, Sec. 6.1-6.2

Assume that on ${}^tS_u$ the displacements are zero (and ${}^tS_u$ is constant). Need to satisfy at time t:

- *Equilibrium* of Cauchy stresses ${}^t\tau_{ij}$ with applied loads

$$ {}^t\tau^T = \left[\ {}^t\tau_{11} \quad {}^t\tau_{22} \quad {}^t\tau_{33} \quad {}^t\tau_{12} \quad {}^t\tau_{23} \quad {}^t\tau_{31}\ \right] \tag{2.1} $$

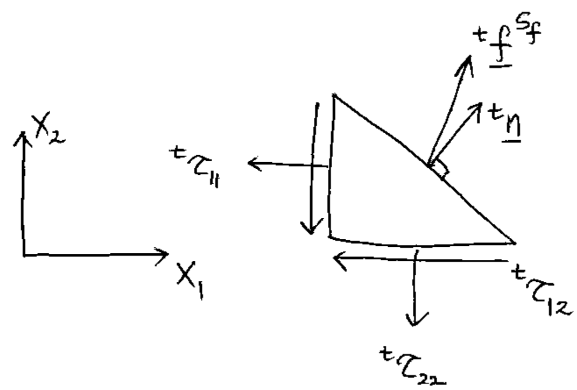
(For $i = 1, 2, 3$)

$$ {}^t\tau_{ij,j} + {}^tf_i^B = 0 \text{ in } {}^tV \text{ (sum over } j) \tag{2.2} $$

$$ {}^t\tau_{ij}\, {}^tn_j = {}^tf_i^{S_f} \text{ on } {}^tS_f \text{ (sum over } j) \tag{2.3} $$

$$ (\text{e.g. } {}^tf_i^{S_f} = {}^t\tau_{i1}\, {}^tn_1 + {}^t\tau_{i2}\, {}^tn_2 + {}^t\tau_{i3}\, {}^tn_3\ ) \tag{2.4} $$

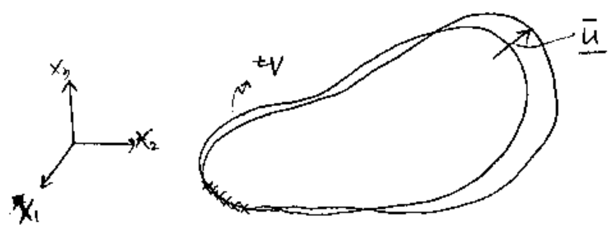
And: ${}^t\tau_{11}\, {}^tn_1 + {}^t\tau_{12}\, {}^tn_2 = {}^tf_1^{S_f}$



- *Compatibility* The displacements ${}^tu_i$ need to be continuous and zero on ${}^tS_u$.
- *Stress-Strain law*

$$ {}^t\tau_{ij} = \text{ function}\left({}^tu_j\right) \tag{2.5} $$

2.1 Principle of Virtual Work*



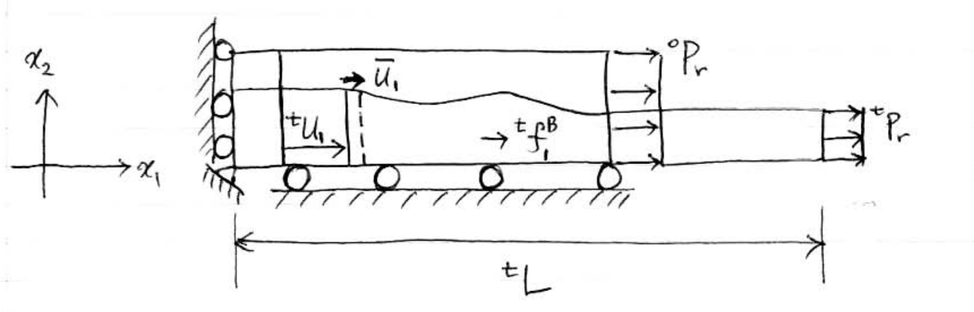
$$\int_{{}^tV} {}^t\tau_{ij}\, {}_t\overline{e}_{ij}\, d\,{}^tV = \int_{{}^tV} {}^tf_i^B\, \overline{u}_i\, d\,{}^tV + \int_{{}^tS_f} {}^tf_i^{S_f}\, \overline{u}_i^{S_f}\, d\,{}^tS_f \tag{2.6}$$

where

$$ {}_t\overline{e}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial\, {}^tx_j} + \frac{\partial \overline{u}_j}{\partial\, {}^tx_i}\right) \tag{2.7}$$

$$\text{with } \overline{u}_i\Big|_{{}^tS_u} = 0 \tag{2.8}$$

2.2 Example



Assume "plane sections remain plane"

Principle of Virtual Work

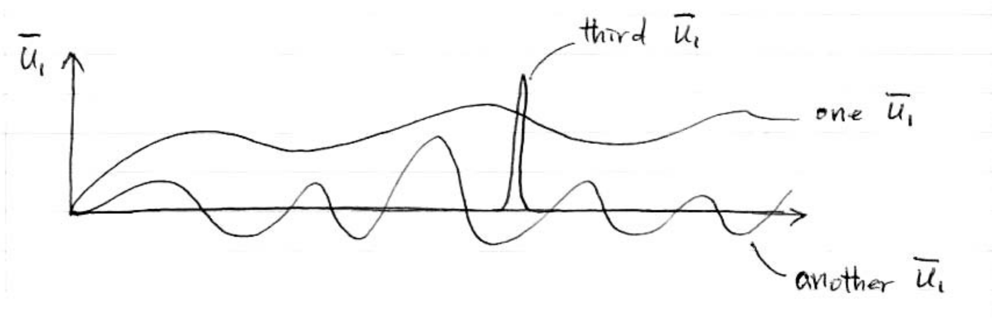
$$\int_{{}^tV} {}^t\tau_{11}\, {}_t\overline{e}_{11}\, d\,{}^tV = \int_{{}^tV} {}^tf_1^B\, \overline{u}_1\, d\,{}^tV + \int_{{}^tS_f} {}^tP_r\, \overline{u}_1^{S_f}\, d\,{}^tS_f \tag{2.9}$$

Derivation of (2.9)

$$ {}^t\tau_{11,1} + {}^tf_1^B = 0 \quad \text{by (2.2)} \tag{2.10}$$

$$\left({}^t\tau_{11,1} + {}^tf_1^B\right)\overline{u}_1 = 0 \tag{2.11}$$

*or Principle of Virtual Displacements



Hence,

$$\int_{{}^tV} \left({}^t\tau_{11,1} + {}^tf_1^B \right) \overline{u}_1 \, d\,{}^tV = 0 \tag{2.12}$$

$$\underbrace{{}^t\tau_{11}\overline{u}_1\Big|_{{}^tS_u}^{{}^tS_f}}_{\overline{u}_1^{S_f}\,{}^t\tau_{11}\,{}^tS_f} - \int_{{}^tV} \underbrace{\overline{u}_{1,1}}_{{}_t\overline{e}_{11}} {}^t\tau_{11} \, d^tV + \int_{{}^tV} \overline{u}_1 \, {}^tf_1^B \, d\,{}^tV = 0 \tag{2.13}$$

where ${}^t\tau_{11}|_{{}^tS_f} = {}^tP_r$.

Therefore we have

$$\int_{{}^tV} {}_t\overline{e}_{11} \, {}^t\tau_{11} \, d\,{}^tV = \int_{{}^tV} \overline{u}_1 \, {}^tf_1^B \, d\,{}^tV + \overline{u}_1^{S_f} \, {}^tP_r \, {}^tS_f \tag{2.14}$$

From (2.12) to (2.14) we simply used mathematics. Hence, if (2.2) and (2.3) are satisfied, then (2.14) must hold. If (2.14) holds, then also (2.2) and (2.3) hold!

Namely, from (2.14)

$$\int_{{}^tV} \overline{u}_{1,1}\,{}^t\tau_{11} d^tV = \overline{u}_1\,{}^t\tau_{11}\Big|_{{}^tS_u}^{{}^tS_f} - \int_{{}^tV} \overline{u}_1\,{}^t\tau_{11,1} d^tV = \int_{{}^tV} \overline{u}_1\,{}^tf_1^B d^tV + \overline{u}_1^{S_f}\,{}^tP_r\,{}^tS_f \tag{2.15}$$

or

$$\int_{{}^tV} \overline{u}_1 \left({}^t\tau_{11,1} + {}^tf_1^B\right) d^tV + \overline{u}_1^{S_f} \left({}^tP_r - {}^t\tau_{11}\right) {}^tS_f = 0 \tag{2.16}$$

Now let $\overline{u}_1 = x\left(1 - \frac{x}{{}^tL}\right)\left({}^t\tau_{11,1} + {}^tf_1^B\right)$, where ${}^tL$ = length of bar.

Hence we must have from (2.16)

$${}^t\tau_{11,1} + {}^tf_1^B = 0 \tag{2.17}$$

and then also

$${}^tP_r = {}^t\tau_{11} \tag{2.18}$$

Lecture 3 - Finite element formulation for solids and structures

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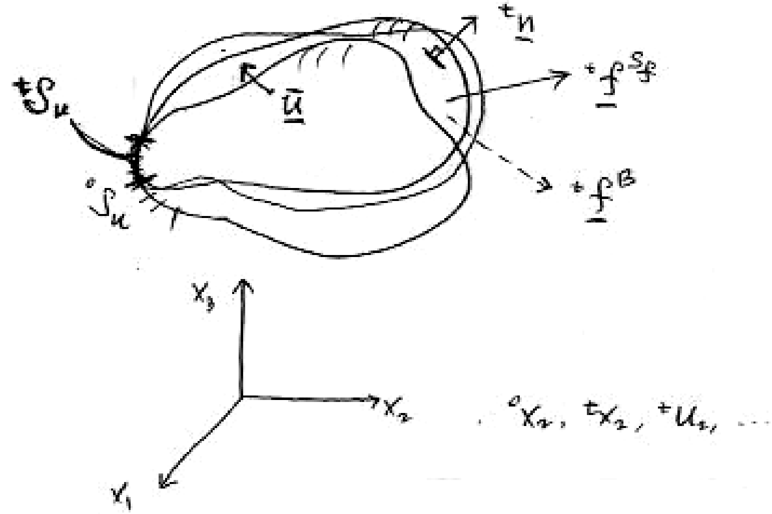


Reading:
Sec. 6.1-6.2

We need to satisfy at time t:

- *Equilibrium*

$$\frac{\partial^t \tau_{ij}}{\partial^t x_j} + {}^t f_i^B = 0 \quad (i = 1,2,3) \text{ in } {}^t V \tag{3.1}$$

$${}^t\tau_{ij}\,{}^t n_j = {}^t f_i^{S_f} \quad (i = 1,2,3) \text{ on } {}^t S_f \tag{3.2}$$

- *Compatibility*
- *Stress-strain law(s)*

Principle of virtual displacements

$$\int_{^tV} {}^t\tau_{ij}\,{}_t\bar{e}_{ij}\; d^tV = \int_{^tV} \bar{u}_i\,{}^t f_i^B\; d^tV + \int_{^tS_f} \bar{u}_i|_{^tS_f}\;{}^t f_i^{S_f}\; d^tS_f \tag{3.3}$$

$${}_t\bar{e}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial\, {}^t x_j} + \frac{\partial \bar{u}_j}{\partial\, {}^t x_i}\right) \tag{3.4}$$

- If (3.3) holds for any continuous virtual displacement (zero on ${}^tS_u$), then (3.1) and (3.2) hold and vice versa.
- Refer to Ex. 4.2 in the textbook.

Major steps

I. Take (3.1) and weigh with $\overline{u}_i$:

$$\left({}^{t}\tau_{ij,j} + {}^{t}f_i^B \right) \overline{u}_i = 0. \tag{3.5a}$$

II. Integrate (3.5a) over volume ${}^{t}V$:

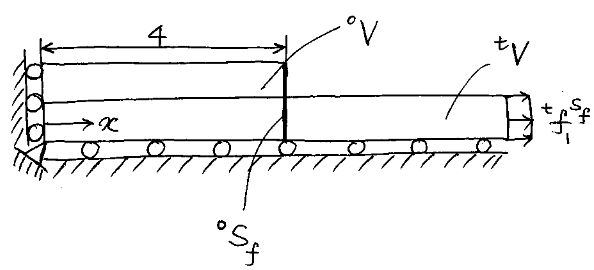
$$\int_{{}^{t}V} \left({}^{t}\tau_{ij,j} + {}^{t}f_i^B \right) \overline{u}_i \, d\,{}^{t}V = 0 \tag{3.5b}$$

III. Use divergence theorem. Obtain a boundary term of stresses times virtual displacements on ${}^{t}S = {}^{t}S_u \cup {}^{t}S_f$.

IV. But, on ${}^{t}S_u$ the $\overline{u}_i = 0$ and on ${}^{t}S_f$ we have (3.2) to satisfy.

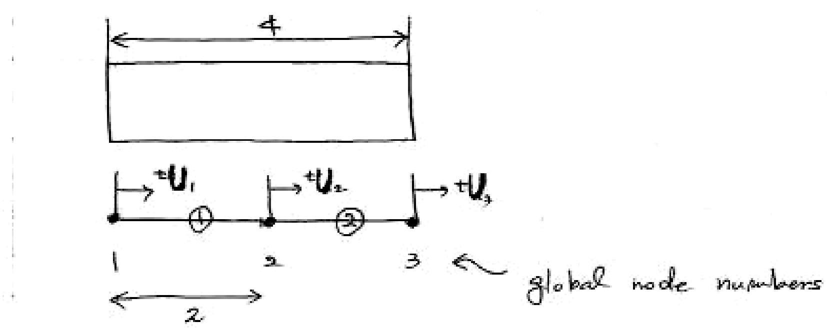
Result: (3.3).

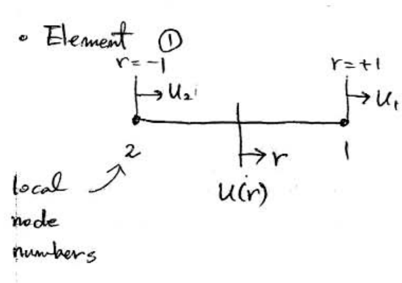
Example



$$\int_{{}^{t}V} {}^{t}\tau_{11}\, {}_{t}\overline{e}_{11} \, d^{t}V = \int_{{}^{t}S_f} \overline{u}_i \, {}^{t}f_1^{S_f} \, d^{t}S_f \tag{3.6}$$

One element solution:





$$u(r) = \frac{1}{2}(1+r)\,u_1 + \frac{1}{2}(1-r)\,u_2 \tag{3.7}$$

$$^{t}u(r) = \frac{1}{2}(1+r)\ {}^{t}u_1 + \frac{1}{2}(1-r)\ {}^{t}u_2 \tag{3.8}$$

$$\overline{u}(r) = \frac{1}{2}(1+r)\,\overline{u}_1 + \frac{1}{2}(1-r)\,\overline{u}_2 \tag{3.9}$$

Suppose we know ${}^{t}\tau_{11}$, ${}^{t}V$, ${}^{t}S_f$, ${}^{t}u$ ... use (3.6).

For element 1,

$${}_{t}\overline{e}_{11} = \frac{\partial \overline{u}}{\partial\, {}^{t}x} = \boldsymbol{B}^{(1)} \begin{pmatrix} \overline{u}_1 \\ \overline{u}_2 \end{pmatrix} \tag{3.10}$$

$$\int_{{}^{t}V} {}_{t}\overline{e}_{11}^{T}{}_{t}\tau_{11} d^{t}V \xrightarrow{\text{for el. (1)}} [\overline{u}_1 \quad \overline{u}_2] \underbrace{\int_{{}^{t}V} \boldsymbol{B}^{(1)^T}\ {}^{t}\tau_{11}\ d\ {}^{t}V}_{=\ {}^{t}\boldsymbol{F}^{(1)}} \tag{3.11}$$

$$\xrightarrow{\text{for el. (1)}} [\overline{u}_1 \quad \overline{u}_2]\ {}^{t}\boldsymbol{F}^{(1)} \tag{3.12}$$

$$= \begin{bmatrix} \underbrace{\overline{U}_1}_{\overline{u}_2} & \underbrace{\overline{U}_2}_{\overline{u}_1} & \overline{U}_3 \end{bmatrix} \begin{bmatrix} {}^{t}\hat{\boldsymbol{F}}^{(1)} \\ 0 \end{bmatrix} \tag{3.13}$$

where

$${}^{t}\hat{F}_1^{(1)} = {}^{t}F_2^{(1)} \tag{3.14}$$

$${}^{t}\hat{F}_2^{(1)} = {}^{t}F_1^{(1)} \tag{3.15}$$

For element 2, similarly,

$$= \begin{bmatrix} \overline{U}_1 & \underbrace{\overline{U}_2}_{\overline{u}_2} & \underbrace{\overline{U}_3}_{\overline{u}_1} \end{bmatrix} \begin{bmatrix} 0 \\ {}^{t}\hat{\boldsymbol{F}}^{(2)} \end{bmatrix} \tag{3.16}$$

R.H.S.

$$\underbrace{\begin{bmatrix} \overline{U}_1 & \overline{U}_2 & \overline{U}_3 \end{bmatrix}}_{\overline{\boldsymbol{U}}^T} \begin{bmatrix} \text{(unknown reaction at left)} \\ 0 \\ {}^{t}S_f \cdot {}^{t}f_1^{S_f} \end{bmatrix} \tag{3.17}$$

Now apply,

$$\overline{\boldsymbol{U}}^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{3.18}$$

then,

$$\overline{\boldsymbol{U}}^T = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \tag{3.19}$$

then,

$$\overline{\boldsymbol{U}}^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \tag{3.20}$$

This gives,

$$\begin{bmatrix} {}^t\hat{\boldsymbol{F}}^{(1)} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ {}^t\hat{\boldsymbol{F}}^{(2)} \end{bmatrix} = \begin{bmatrix} \text{unknown reaction} \\ 0 \\ {}^tf_1^{{}^tS_f} \cdot {}^tS_f \end{bmatrix} \tag{3.21}$$

We write that as

$$ {}^t\boldsymbol{F} = {}^t\boldsymbol{R} \tag{3.22}$$

$$ {}^t\boldsymbol{F} = \text{fn}\left({}^tU_1,\ {}^tU_2,\ {}^tU_3 \right) \tag{3.23}$$

Lecture 4 - Finite element formulation for solids and structures

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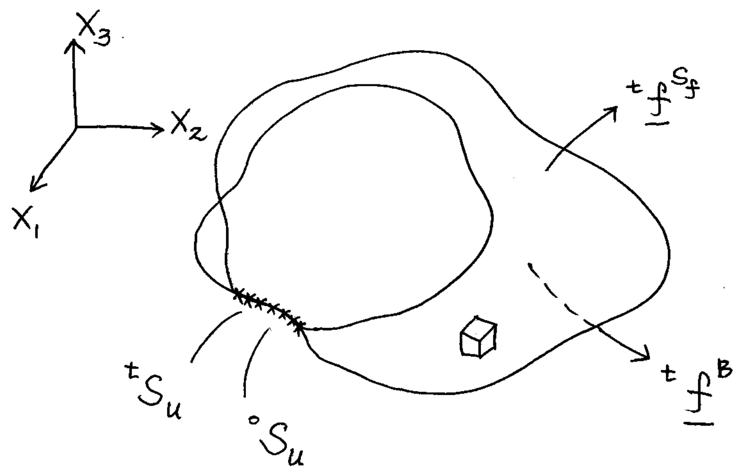
We considered a general 3D body,

Reading: Ch. 4



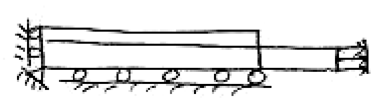
The exact solution of the mathematical model must satisfy the conditions:

- *Equilibrium* within ${}^tV$ and on ${}^tS_f$,
- *Compatibility*
- *Stress-strain law(s)*

I. Differential formulation

II. Variational formulation (Principle of virtual displacements) (or weak formulation)

We developed the governing F.E. equations for a sheet or bar



We obtained

$$\boxed{{}^t\boldsymbol{F} = {}^t\boldsymbol{R}} \tag{4.1}$$

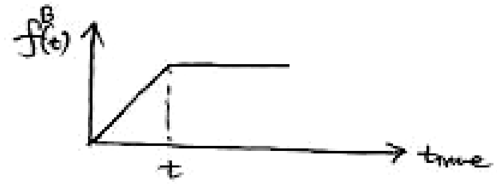
where ${}^t\boldsymbol{F}$ is a function of displacements/stresses/material law; and ${}^t\boldsymbol{R}$ is a function of time.

Assume for now linear analysis: Equilibrium within ${}^0V$ and on ${}^0S_f$, linear stress-strain law and small displacements yields

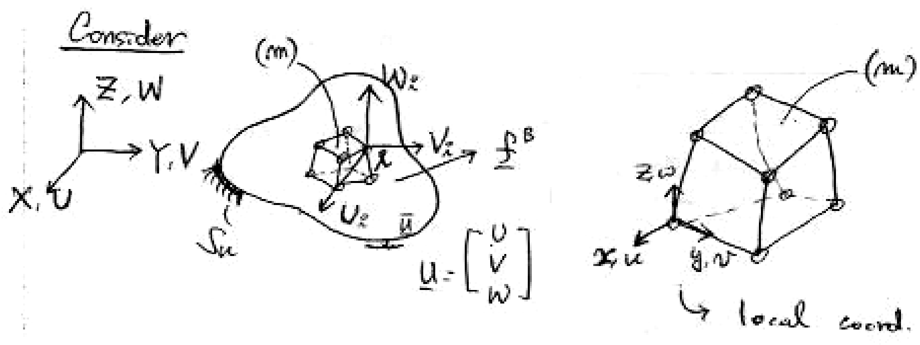
$$ {}^t\boldsymbol{F} = \boldsymbol{K} \cdot {}^t\boldsymbol{U} \tag{4.2}$$

We want to establish,

$$\boldsymbol{K}\boldsymbol{U}(t) = \boldsymbol{R}(t) \tag{4.3}$$



Consider



$$\hat{\boldsymbol{U}}^T = \begin{bmatrix} U_1 & V_1 & W_1 & U_2 & \cdots & W_N \end{bmatrix} \quad (N \text{ nodes}) \tag{4.4}$$

where $\hat{\boldsymbol{U}}^T$ is a distinct nodal point displacement vector.

> **Note:** for the moment "remove S_u"

We also say

$$\hat{\boldsymbol{U}}^T = \begin{bmatrix} U_1 & U_2 & U_3 & \cdots & U_n \end{bmatrix} \quad (n = 3N) \tag{4.5}$$

We now assume

$$\boldsymbol{u}^{(\boldsymbol{m})} = \boldsymbol{H}^{(\boldsymbol{m})}\hat{\boldsymbol{U}}, \quad \boldsymbol{u}^{(\boldsymbol{m})} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}^{(m)} \tag{4.6a}$$

where $\boldsymbol{H}^{(\boldsymbol{m})}$ is 3 x n and $\hat{\boldsymbol{U}}$ is n x 1.

$$\boldsymbol{\epsilon}^{(\boldsymbol{m})} = \boldsymbol{B}^{(\boldsymbol{m})}\hat{\boldsymbol{U}} \tag{4.6b}$$

where $\boldsymbol{B}^{(\boldsymbol{m})}$ is 6 x n, and

$$\boldsymbol{\epsilon}^{(m)^T} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{yy} & \epsilon_{zz} & \gamma_{xy} & \gamma_{yz} & \gamma_{zx} \end{bmatrix}$$

e.g. $\gamma_{xy} = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}$

We also assume

$$\overline{\boldsymbol{u}}^{(m)} = \boldsymbol{H}^{(m)}\overline{\hat{\boldsymbol{U}}} \tag{4.6c}$$

$$\overline{\boldsymbol{\epsilon}}^{(m)} = \boldsymbol{B}^{(m)}\overline{\hat{\boldsymbol{U}}} \tag{4.6d}$$

Principle of Virtual Work:

$$\int_V \overline{\boldsymbol{\epsilon}}^T \boldsymbol{\tau} dV = \int_V \overline{\hat{\boldsymbol{U}}}^T \boldsymbol{f}^B dV \tag{4.7}$$

(4.7) can be rewritten as

$$\sum_m \int_{V^{(m)}} \overline{\boldsymbol{\epsilon}}^{(m)^T} \boldsymbol{\tau}^{(m)} dV^{(m)} = \sum_m \int_{V^{(m)}} \overline{\hat{\boldsymbol{U}}}^{(m)^T} \boldsymbol{f}^{B^{(m)}} dV^{(m)} \tag{4.8}$$

Substitute (4.6a) to (4.6d).

$$\begin{aligned} &\overline{\hat{\boldsymbol{U}}}^T \left\{ \sum_m \int_{V^{(m)}} \boldsymbol{B}^{(m)^T} \boldsymbol{\tau}^{(m)} \, dV^{(m)} \right\} = \\ &\quad \overline{\hat{\boldsymbol{U}}}^T \left\{ \sum_m \int_{V^{(m)}} \boldsymbol{H}^{(m)^T} \boldsymbol{f}^{B^{(m)}} \, dV^{(m)} \right\} \end{aligned} \tag{4.9}$$

$$\boldsymbol{\tau}^{(m)} = \boldsymbol{C}^{(m)} \boldsymbol{\epsilon}^{(m)} = \boldsymbol{C}^{(m)} \boldsymbol{B}^{(m)} \hat{\boldsymbol{U}} \tag{4.10}$$

Finally,

$$\begin{aligned} &\cancel{\overline{\hat{\boldsymbol{U}}}^T} \left\{ \sum_m \int_{V^{(m)}} \boldsymbol{B}^{(m)^T} \boldsymbol{C}^{(m)} \boldsymbol{B}^{(m)} \, dV^{(m)} \right\} \hat{\boldsymbol{U}} = \\ &\quad \cancel{\overline{\hat{\boldsymbol{U}}}^T} \left\{ \sum_m \int_{V^{(m)}} \boldsymbol{H}^{(m)^T} \boldsymbol{f}^{B^{(m)}} \, dV^{(m)} \right\} \end{aligned} \tag{4.11}$$

with

$$\overline{\boldsymbol{\epsilon}}^{(m)^T} = \overline{\hat{\boldsymbol{U}}}^T \boldsymbol{B}^{(m)^T} \tag{4.12}$$

$$\boldsymbol{K} \hat{\boldsymbol{U}} = \boldsymbol{R}_B \tag{4.13}$$

where $\boldsymbol{K}$ is n x n, and $\boldsymbol{R}_B$ is n x 1.

Direct stiffness method:

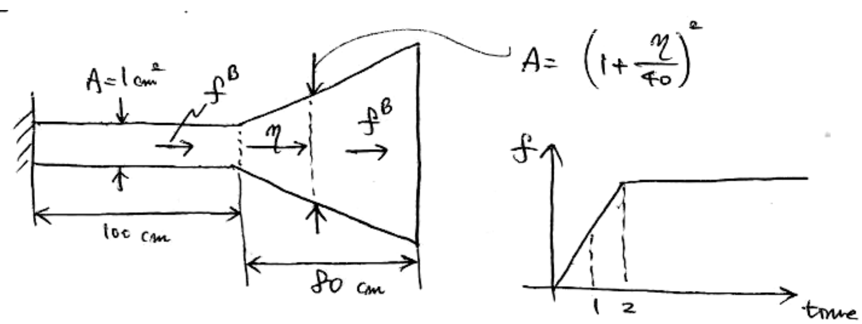
$$\boldsymbol{K} = \sum_m \boldsymbol{K}^{(m)} \tag{4.14}$$

$$\boldsymbol{R}_B = \sum_m \boldsymbol{R}_B^{(m)} \tag{4.15}$$

$$\boldsymbol{K}^{(m)} = \int_{V^{(m)}} \boldsymbol{B}^{(m)^T} \boldsymbol{C}^{(m)} \boldsymbol{B}^{(m)} \, dV^{(m)} \tag{4.16}$$

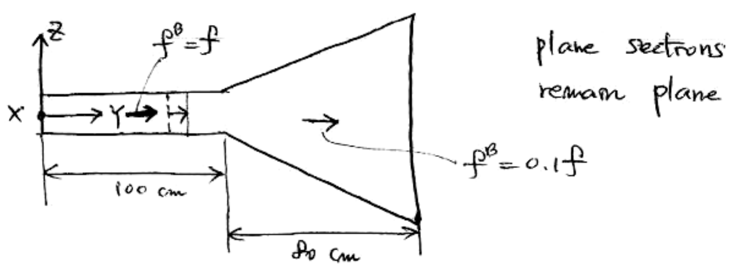
$$\boldsymbol{R}_B^{(m)} = \int_{V^{(m)}} \boldsymbol{H}^{(m)^T} \boldsymbol{f}^{B^{(m)}} \, dV^{(m)} \tag{4.17}$$

Example 4.5 textbook

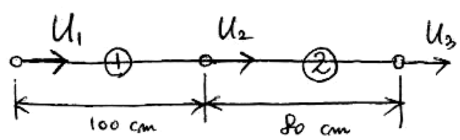


E = Young's Modulus

Mathematical model Plane sections remain plane:

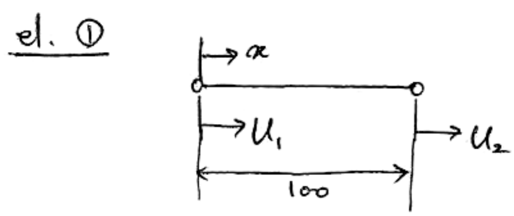


F.E. model

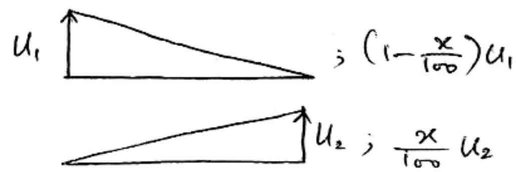


$$\boldsymbol{U} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \tag{4.18}$$

Element 1



$$u^{(1)}(x) = \underbrace{\begin{bmatrix} 1 - \frac{x}{100} & \frac{x}{100} & 0 \end{bmatrix}}_{\boldsymbol{H}^{(1)}} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \tag{4.19}$$



$$\epsilon_{xx}^{(1)}(x) = \underbrace{\begin{bmatrix} -\frac{1}{100} & \frac{1}{100} & 0 \end{bmatrix}}_{\boldsymbol{B}^{(1)}} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \tag{4.20}$$

Element 2

$$u^{(2)}(x) = \underbrace{\begin{bmatrix} 0 & 1-\frac{x}{80} & \frac{x}{80} \end{bmatrix}}_{\boldsymbol{H}^{(2)}} \boldsymbol{U} \tag{4.21}$$

$$\epsilon_{xx}^{(2)}(x) = \underbrace{\begin{bmatrix} 0 & -\frac{1}{80} & \frac{1}{80} \end{bmatrix}}_{\boldsymbol{B}^{(2)}} \boldsymbol{U} \tag{4.22}$$

Then,

$$\boldsymbol{K} = \frac{E}{100} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{13E}{240} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{4.23}$$

where,

$$\frac{E(1)}{100} \equiv \left(\frac{AE}{L}\right) \tag{4.24}$$

$$\frac{E \cdot 13}{3 \cdot 80} = \underbrace{\left(\frac{13}{3}\right)}_{A^*} \frac{E}{80} \tag{4.25}$$

$$\begin{array}{ccccc} A\Big|_{\eta=0} & < & A^* & < & A\Big|_{\eta=80} \\ 1 & < & 4.333 & < & 9 \end{array} \tag{4.26}$$

Lecture 5 - F.E. displacement formulation, cont'd

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For the continuum

Reading: Ch. 4

- Differential formulation
- Variational formulation (Principle of Virtual Displacements)

Next, we assumed infinitesimal small displacement, Hooke's Law, linear analysis

$$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{R} \tag{5.1a}$$

$$\boxed{\boldsymbol{u}^{(m)} = \boldsymbol{H}^{(m)}\boldsymbol{U}} \tag{5.1b}$$

$$\boldsymbol{K} = \sum_m \boldsymbol{K}^{(m)} \tag{5.1c}$$

$$\boldsymbol{R} = \sum_m \boldsymbol{R}_B^{(m)} \tag{5.1d}$$

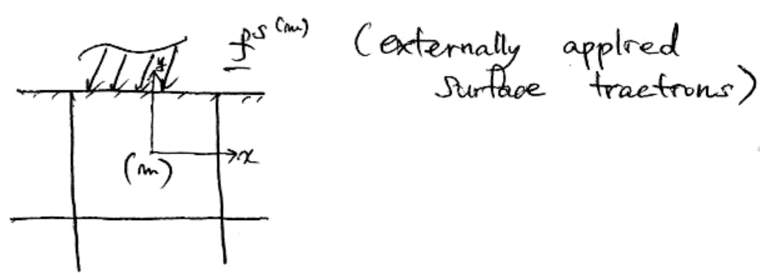
$$\boldsymbol{\epsilon}^{(m)} = \boldsymbol{B}^{(m)}\boldsymbol{U} \tag{5.1e}$$

$$\boldsymbol{U}^T = \begin{bmatrix} U_1 & U_2 & \cdots & U_n \end{bmatrix},\ (n = \text{all d.o.f. of element assemblage}) \tag{5.1f}$$

$$\boldsymbol{K}^{(m)} = \int_{V^{(m)}} \boldsymbol{B}^{(m)^T}\, \boldsymbol{C}^{(m)}\, \boldsymbol{B}^{(m)}\, dV^{(m)} \tag{5.1g}$$

$$\boldsymbol{R}_B^{(m)} = \int_{V^{(m)}} \boldsymbol{H}^{(m)^T}\, \boldsymbol{f}^{B^{(m)}}\, dV^{(m)} \tag{5.1h}$$

Surface loads



Recall that in the principle of virtual displacements,

$$\text{“surface” loads} = \int_{S_f} \overline{\boldsymbol{U}}^{S_f^T}\, \boldsymbol{f}^{S_f}\, dS_f \tag{5.2}$$

$$\boldsymbol{u}^{S(m)} = \boldsymbol{H}^{S(m)}\boldsymbol{U} \tag{5.3}$$

$$\boldsymbol{H}^{S(m)} = \boldsymbol{H}^{(m)}\Big|_{\text{evaluated at the surface}} \tag{5.4}$$

Substitute into (5.2)

$$\overline{\boldsymbol{U}}^T \int_{S^{(m)}} \boldsymbol{H}^{S^{(m)}\,T} \boldsymbol{f}^{S^{(m)}}\, dS^{(m)} \tag{5.5}$$

for element (m) and one surface of that element.

$$\boldsymbol{R}_s^{(m)} = \int_{S^{(m)}} \boldsymbol{H}^{S^{(m)}\,T} \boldsymbol{f}^{S^{(m)}}\, dS^{(m)} \tag{5.6}$$

Need to add contributions from all surfaces of all loaded external elements.

$$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{R}_B + \boldsymbol{R}_S + \boldsymbol{R}_c \tag{5.7}$$

where $\boldsymbol{R}_c$ are concentrated nodal loads.

Assume

- (5.7) has been established without any displacement boundary conditions.
- We, however, know nodal displacements $\boldsymbol{U}_b$ (rewriting (5.7)).

$$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{R} \Rightarrow \begin{bmatrix} \boldsymbol{K}_{aa} & \boldsymbol{K}_{ab} \\ \boldsymbol{K}_{ba} & \boldsymbol{K}_{bb} \end{bmatrix} \begin{pmatrix} \boldsymbol{U}_a \\ \boldsymbol{U}_b \end{pmatrix} = \begin{pmatrix} \boldsymbol{R}_a \\ \boldsymbol{R}_b \end{pmatrix} \tag{5.8}$$

Solve for $\boldsymbol{U}_a$:

$$\boldsymbol{K}_{aa}\boldsymbol{U}_a = \boldsymbol{R}_a - \boldsymbol{K}_{ab}\boldsymbol{U}_b \tag{5.9}$$

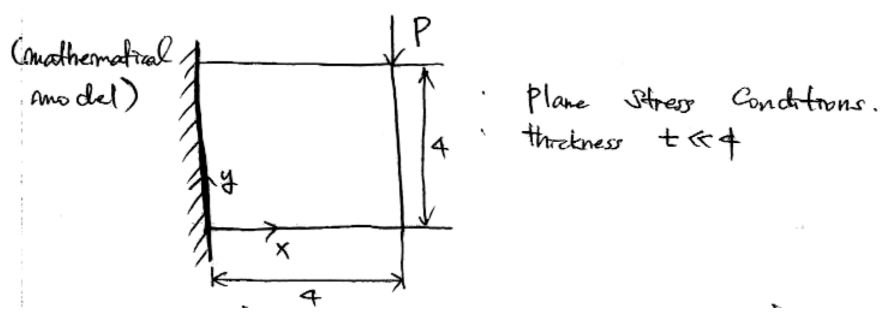
where $\boldsymbol{U}_b$ is known!

Then use

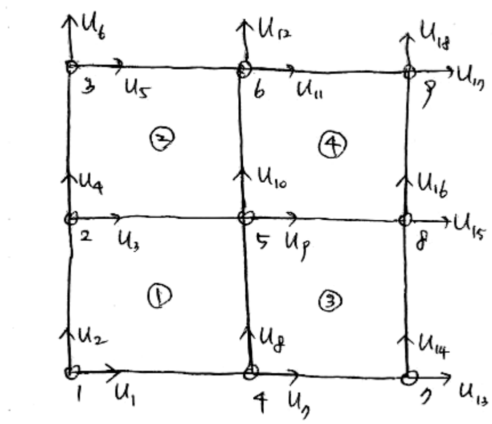
$$\boldsymbol{K}_{ba}\boldsymbol{U}_a + \boldsymbol{K}_{bb}\boldsymbol{U}_b = \boldsymbol{R}_b + \boldsymbol{R}_r \tag{5.10}$$

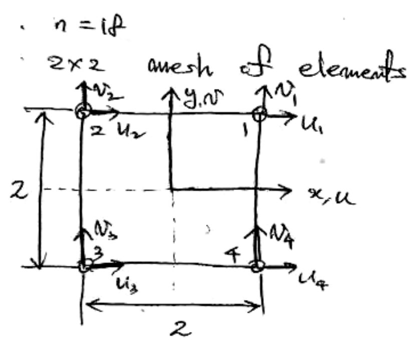
where $\boldsymbol{R}_r$ are unknown reactions.

Example 4.6 textbook



$$\begin{pmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{pmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix} \tag{5.11}$$





$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \boldsymbol{H} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \tag{5.12}$$

If we can set this relation up, then clearly we can get $\boldsymbol{H}^{(1)}$, $\boldsymbol{H}^{(2)}$, $\boldsymbol{H}^{(3)}$, $\boldsymbol{H}^{(4)}$.

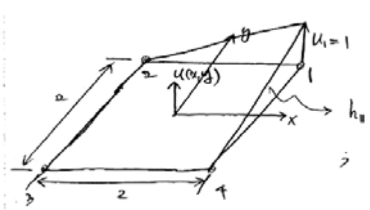
$$\boldsymbol{u}^{(m)} = \boldsymbol{H}^{(m)}\boldsymbol{U} \tag{5.13}$$

Also want $\boldsymbol{\epsilon}^{(m)} = \boldsymbol{B}^{(m)}\boldsymbol{U}$. *We want* $\boldsymbol{H}$. We could proceed this way

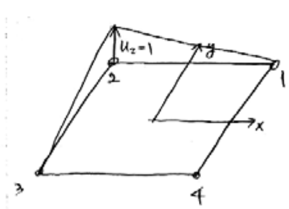
$$u(x,y) = a_1 + a_2x + a_3y + a_4xy \tag{5.14}$$
$$v(x,y) = b_1 + b_2x + b_3y + b_4xy \tag{5.15}$$

Express $a_1 \ldots a_4$, $b_1 \ldots b_4$ in terms of the nodal displacements $u_1 \ldots u_4$, $v_1 \ldots v_4$.

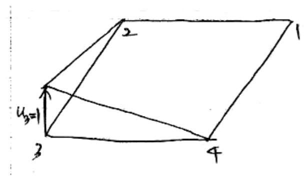
(e.g.) $u(1,1) = a_1 + a_2 + a_3 + a_4 = u_1$.



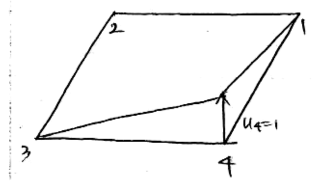
$h_1(x,y) = \frac{1}{4}(1+x)(1+y)$ interpolation function for node 1.



$h_2(x,y) = \frac{1}{4}(1-x)(1+y)$



$$h_3(x,y) = \tfrac{1}{4}(1-x)(1-y)$$



$$h_4(x,y) = \tfrac{1}{4}(1+x)(1-y)$$

$$\boxed{u(x,y) = h_1u_1 + h_2u_2 + h_3u_3 + h_4u_4} \tag{5.16}$$

$$\boxed{v(x,y) = h_1v_1 + h_2v_2 + h_3v_3 + h_4v_4} \tag{5.17}$$

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \underbrace{\begin{bmatrix} h_1 & h_2 & h_3 & h_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_1 & h_2 & h_3 & h_4 \end{bmatrix}}_{\boldsymbol{H}\ (2\text{x}8)} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \tag{5.18}$$

We also want,

$$\begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \underbrace{\begin{bmatrix} h_{1,x} & h_{2,x} & h_{3,x} & h_{4,x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_{1,y} & h_{2,y} & h_{3,y} & h_{4,y} \\ h_{1,y} & h_{2,y} & h_{3,y} & h_{4,y} & h_{1,x} & h_{2,x} & h_{3,x} & h_{4,x} \end{bmatrix}}_{\boldsymbol{B}\ (3\text{x}8)} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \tag{5.19}$$

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \tag{5.20}$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} \tag{5.21}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{5.22}$$

2.094 — Finite Element Analysis of Solids and Fluids Fall '08

Lecture 6 - Finite element formulation, example, convergence

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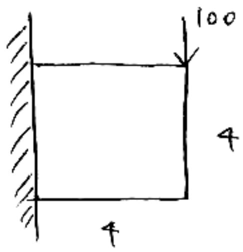
6.1 Example

Reading: Ex. 4.6 in the text



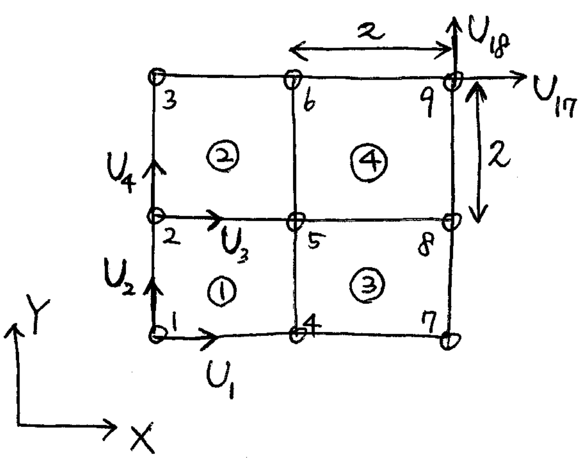
$t = 0.1, \quad E, \nu$ plane stress

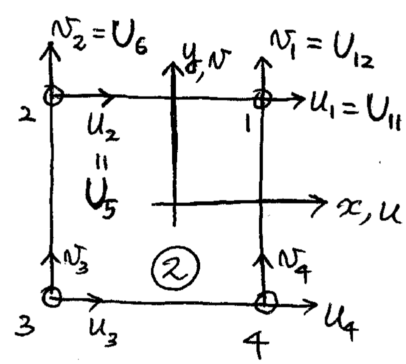
$$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{R}; \quad \boldsymbol{R} = \boldsymbol{R}_B + \boldsymbol{R}_s + \boldsymbol{R}_c + \boldsymbol{R}_r \tag{6.1}$$

$$\boldsymbol{K} = \sum_m \boldsymbol{K}^{(m)}; \quad \boldsymbol{K}^{(m)} = \int_{V^{(m)}} \boldsymbol{B}^{(m)^T} \boldsymbol{C}^{(m)} \boldsymbol{B}^{(m)} \, d\,V^{(m)} \tag{6.2}$$

$$\boldsymbol{R}_B = \sum_m \boldsymbol{R}_B^{(m)}; \quad \boldsymbol{R}_B^{(m)} = \int_{V^{(m)}} \boldsymbol{H}^{(m)^T} \boldsymbol{f}^{B(m)} \, d\,V^{(m)} \tag{6.3}$$

6.1.1 F.E. model





$$\boldsymbol{K}\Big|_{\text{el. (2)}} = \begin{array}{c} \begin{array}{cccccccc} u_1 & u_2 & u_3 & u_4 & v_1 & v_2 & v_3 & v_4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} \bigcirc & \square & \triangle & \times & \times & \times & \times & \times \\ \vdots & & & & & & & \\ & & & & & & & \end{bmatrix} \end{array} \begin{array}{c} \\ \leftarrow u_1 \\ \vdots \\ \\ \end{array} \tag{6.4}$$

In practice,

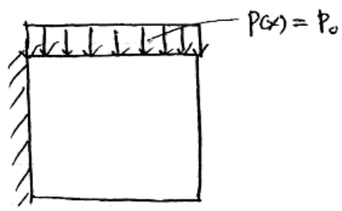
$$\boldsymbol{K}\Big|_{\text{el}} = \int_V \boldsymbol{B}^T \boldsymbol{C} \boldsymbol{B}\, dV; \quad \boldsymbol{\epsilon} = \boldsymbol{B} \begin{pmatrix} u_1 \\ \vdots \\ u_4 \\ v_1 \\ \vdots \\ v_4 \end{pmatrix} \tag{6.5}$$

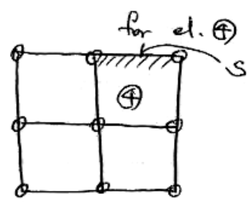
where $\boldsymbol{K}$ is 8x8 and $\boldsymbol{B}$ is 3x8.

Assume we have $\boldsymbol{K}$ (8x8) for el. (2)

$$\underbrace{\boldsymbol{K}}_{\text{assemblage}} = \begin{array}{c} \begin{array}{ccccccccc} U_1 & U_2 & U_3 & U_4 & U_5 & \cdots & U_{11} & \cdots & U_{18} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \left[\begin{array}{ccccccccc} \times & \times & \times & \times & \times & \times & \times & \times & \times \\ & & \vdots & & \vdots & & \vdots & & \\ & & \vdots & & \vdots & & \vdots & & \\ & & \triangle & & \square & & \bigcirc & & \\ & & \vdots & & \vdots & & \vdots & & \\ & & \vdots & & \vdots & & \vdots & & \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \end{array} \right] \end{array} \begin{array}{l} \leftarrow U_1 \\ \quad\vdots \\ \quad\vdots \\ \leftarrow U_{11} \\ \quad\vdots \\ \quad\vdots \\ \leftarrow U_{18} \end{array} \tag{6.6}$$

Consider,





$$\boldsymbol{R}_S = \int_S \boldsymbol{H}^{S^T} \boldsymbol{f}^S \, d\, S; \quad \boldsymbol{H}^S = \boldsymbol{H}\Big|_{\text{on surface}} \tag{6.7}$$

$$\boldsymbol{H} = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_1 & h_2 & h_3 & h_4 \end{bmatrix} \begin{array}{l} \leftarrow u(x,y) \\ \leftarrow v(x,y) \end{array} \tag{6.8}$$

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_4 \\ v_1 \\ \vdots \\ v_4 \end{pmatrix} \tag{6.9}$$

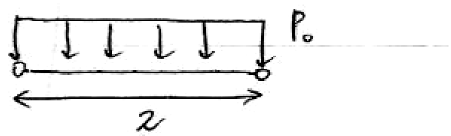
$$\boldsymbol{H}^S = \boldsymbol{H}\Big|_{y=+1} \tag{6.10}$$

$$= \begin{bmatrix} \frac{1}{2}(1+x) & \frac{1}{2}(1-x) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1+x) & \frac{1}{2}(1-x) & 0 & 0 \end{bmatrix} \tag{6.11}$$

From (6.7);

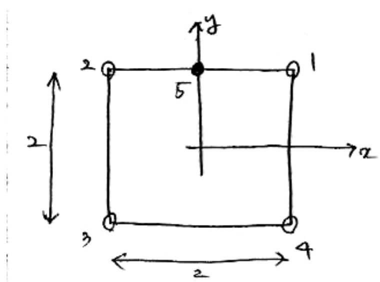
$$\boldsymbol{R}_S = \int_{-1}^{+1} \left[\begin{array}{c|c} \frac{1}{2}(1+x) & 0 \\ \frac{1}{2}(1-x) & 0 \\ 0 & \frac{1}{2}(1+x) \\ 0 & \frac{1}{2}(1-x) \end{array}\right] \begin{bmatrix} 0 \\ -p(x) \end{bmatrix} \underbrace{(0.1)}_{\text{thickness}} \; dx \tag{6.12}$$

$$\boldsymbol{R}_S = \begin{bmatrix} 0 \\ 0 \\ -p_0(0.1) \\ -p_0(0.1) \end{bmatrix} \tag{6.13}$$



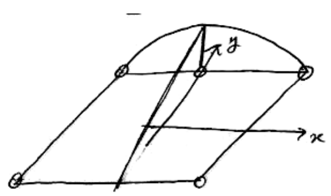
total load $= P_0 \times 0.1 \times 2$

6.1.2 Higher-order elements



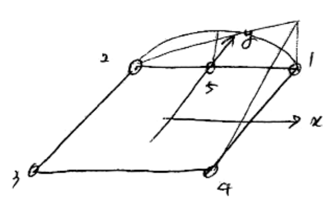
Want h_1, h_2, h_3, h_4, h_5

$u(x,y) = \sum_{i=1}^{5} h_i u_i$.



$h_i = 1$ at node i and 0 at all other nodes.

$h_5 = \frac{1}{2}(1-x^2)(1+y)$



$$h_1 = \frac{1}{4}(1+x)(1+y) - \frac{1}{2}h_5 \tag{6.14}$$

$$h_2 = \frac{1}{4}(1-x)(1+y) - \frac{1}{2}h_5 \tag{6.15}$$

$$h_3 = \frac{1}{4}(1-x)(1-y) \tag{6.16}$$

$$h_1 = \frac{1}{4}(1+x)(1-y) \tag{6.17}$$

Note:

$$\boxed{\sum h_i = 1}$$

We must have $\sum_i h_i = 1$ to satisfy the rigid body mode condition.

$$u(x,y) = \sum_i h_i u_i \tag{6.18}$$

Assume all nodal point displacements $= u^*$. Then,

$$u(x,y) = \sum_i h_i u^* = u^* \sum_i h_i = u^* \tag{6.19}$$

From (6.1),

$$\left(\sum_m \boldsymbol{K}^{(m)}\right)\boldsymbol{U} = \boldsymbol{R} \tag{6.20}$$

$$\sum_m \left[\int_{V^{(m)}} \boldsymbol{B}^{(m)^T} \boldsymbol{C}^{(m)} \boldsymbol{B}^{(m)} dV^{(m)}\right]\boldsymbol{U} = \boldsymbol{R} \tag{6.21}$$

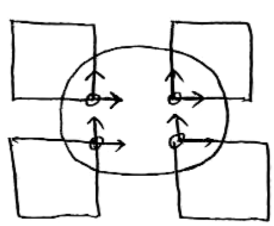
where $\boldsymbol{C}^{(m)}\boldsymbol{B}^{(m)}\boldsymbol{U} = \boldsymbol{\tau}^{(m)}$. (Assume we calculated $\boldsymbol{U}$.)

$$\sum_m \int_{V^{(m)}} \boldsymbol{B}^{(m)^T} \boldsymbol{\tau}^{(m)}\ d\,V^{(m)} = \boldsymbol{R} \tag{6.22}$$

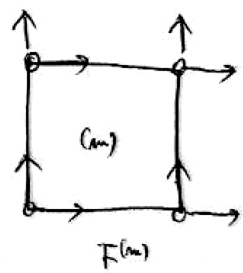
$$\sum_m \boldsymbol{F}^{(m)} = \boldsymbol{R}; \quad \boldsymbol{F}^{(m)} = \int_{V^{(m)}} \boldsymbol{B}^{(m)^T} \boldsymbol{\tau}^{(m)}\ d\,V^{(m)} \tag{6.23}$$

Two properties

I. The sum of the $\boldsymbol{F}^{(m)}$'s at any node is equal to the applied external forces.



II. Every element is in equilibrium under its $\boldsymbol{F}^{(m)}$



$$\hat{\overline{\boldsymbol{U}}}^T \boldsymbol{F}^{(m)} = \underbrace{\hat{\overline{\boldsymbol{U}}}^T \int_{V^{(m)}} \boldsymbol{B}^{(m)^T}}_{=\overline{\boldsymbol{\epsilon}}^{(m)^T}} \boldsymbol{\tau}^{(m)} \, d\, V^{(m)} \tag{6.24}$$

$$= \int_{V^{(m)}} \overline{\boldsymbol{\epsilon}}^{(m)^T} \boldsymbol{\tau}^{(m)} \, d\, V^{(m)} \tag{6.25}$$

$$= 0 \tag{6.26}$$

where $\hat{\overline{\boldsymbol{U}}}^T$ = virtual nodal point displacement.

Apply rigid body displacement.

If we move the element virtually in the rigid body modes, $\overline{\boldsymbol{\epsilon}}^{(m)}$ is zero. Therefore the virtual work obtained due to virtual motion of the element is zero. Then the element is in equilibrium under its $\boldsymbol{F}^{(m)}$.

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Lecture 7 - Isoparametric elements

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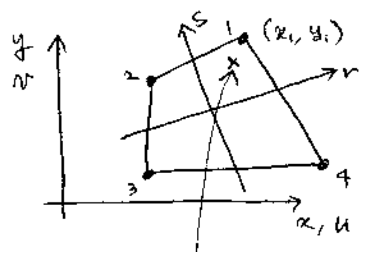
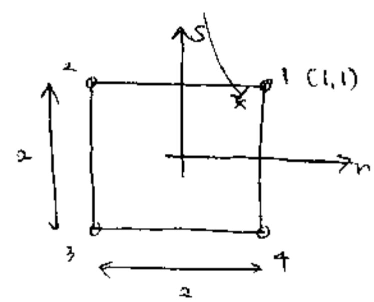


Reading: Sec. 5.1-5.3

We want $\boldsymbol{K} = \int_V \boldsymbol{B}^T \boldsymbol{C} \boldsymbol{B}\, dV$, $\boldsymbol{R}_B = \int_V \boldsymbol{H}^T \boldsymbol{f}^B\, dV$. Unique correspondence $(x, y) \Leftrightarrow (r, s)$



(r, s) are natural coordinate system or isoparametric coordinate system.

$$x = \sum_{i=1}^{4} h_i x_i \tag{7.1}$$

$$y = \sum_{i=1}^{4} h_i y_i \tag{7.2}$$

where

$$h_1 = \frac{1}{4}(1+r)(1+s) \tag{7.3}$$

$$h_2 = \frac{1}{4}(1-r)(1+s) \tag{7.4}$$

$\ldots$

$$u(r,s) = \sum_{i=1}^{4} h_i u_i \tag{7.5}$$

$$v(r,s) = \sum_{i=1}^{4} h_i v_i \tag{7.6}$$

$$\boldsymbol{\epsilon} = \boldsymbol{B}\hat{\boldsymbol{u}} \quad \hat{\boldsymbol{u}}^T = \begin{bmatrix} u_1 & u_2 & \cdots & v_4 \end{bmatrix} \tag{7.7}$$

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \boldsymbol{B}\hat{\boldsymbol{u}} \tag{7.8}$$

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{pmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix}}_{\boldsymbol{J}} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \tag{7.9}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \boldsymbol{J}^{-1} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \end{pmatrix} \tag{7.10}$$

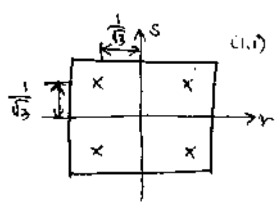
$\boldsymbol{J}$ must be non-singular which ensures that there is unique correspondence between (x, y) and (r, s). Hence,

$$\boldsymbol{K} = \int_{-1}^{1}\int_{-1}^{1} \boldsymbol{B}^T\boldsymbol{C}\boldsymbol{B}\ \underbrace{t\det(\boldsymbol{J})\ dr\ ds}_{d\,V} \tag{7.11}$$

$$\text{Also, } \boldsymbol{R}_B = \int_{-1}^{1}\int_{-1}^{1} \boldsymbol{H}^T\boldsymbol{f}^B\ t\det(\boldsymbol{J})\ dr\ ds \tag{7.12}$$

Numerical integration (Gauss formulae) (Ch. 5.5)

$$\boldsymbol{K} \cong t\sum_i\sum_j \boldsymbol{B}_{ij}^T\boldsymbol{C}\boldsymbol{B}_{ij}\ \det(\boldsymbol{J}_{ij}) \times (\text{weight } i,\ j) \tag{7.13}$$

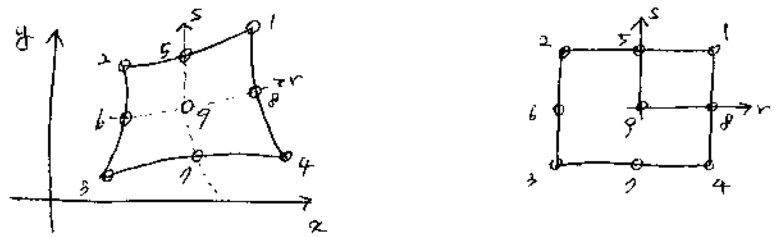


2x2 Gauss integration,

$$(i = 1, 2) \tag{7.14}$$

$$(j = 1, 2) \quad (\text{weight } i, j = 1 \text{ in this case}) \tag{7.15}$$

9-node element



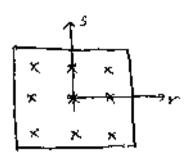
$$x = \sum_{i=1}^{9} h_i x_i \tag{7.16}$$

$$y = \sum_{i=1}^{9} h_i y_i \tag{7.17}$$

$$u = \sum_{i=1}^{9} h_i u_i \tag{7.18}$$

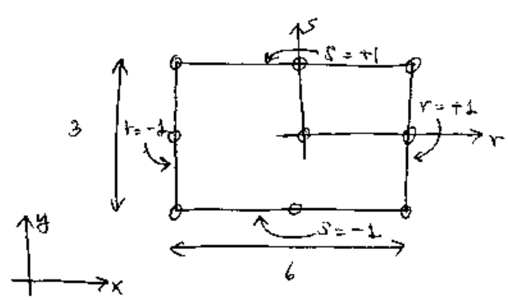
$$v = \sum_{i=1}^{9} h_i v_i \tag{7.19}$$

Use 3x3 Gauss integration



For rectangular elements, $\boldsymbol{J} = \text{const}$

Consider the following element,



Note, here we could use $h_i(x, y)$ directly.

$$\boldsymbol{J} = \begin{bmatrix} 3\left(=\frac{6}{2}\right) & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \tag{7.20}$$

Then, we can determine the number of appropriate integration points by investigating the maximum order of $\boldsymbol{B}^T\boldsymbol{C}\boldsymbol{B}$.

For a rectangular element, 3x3 Gauss integration gives exact $\boldsymbol{K}$ matrix. If the element is distorted, a $\boldsymbol{K}$ matrix which is still accurate enough will be obtained, (if high enough integration is used).

Convergence *Principle of virtual work:*

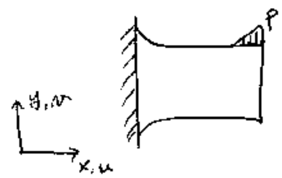
Reading: Sec. 5.5.5, 4.3

$$\int_V \overline{\boldsymbol{\epsilon}}^T \boldsymbol{C} \boldsymbol{\epsilon} \, dV = \mathcal{R}(\overline{\boldsymbol{u}}) \tag{7.21}$$

Find $\boldsymbol{u}$, solution, in V, vector space (any continuous function that satisfies boundary conditions), satisfying

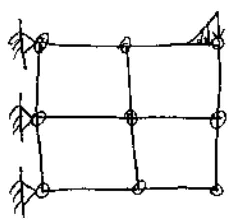
$$\int_V \overline{\boldsymbol{\epsilon}}^T \boldsymbol{C} \boldsymbol{\epsilon} \, dV = \underbrace{a(\boldsymbol{u}, \boldsymbol{v})}_{\text{bilinear form}} = \underbrace{(\boldsymbol{f}, \boldsymbol{v})}_{\mathcal{R}(\boldsymbol{v})} \quad \text{for all } \boldsymbol{v}\text{, an element of V.} \tag{7.22}$$

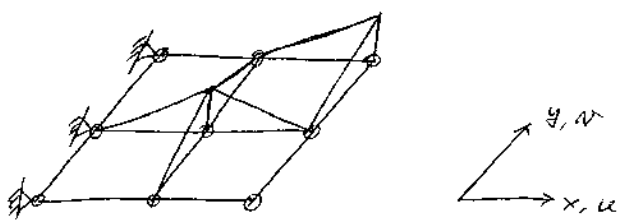
Example:



Finite Element problem Find $\boldsymbol{u}_h \in \mathrm{V}_h$, where V_h is F.E. vector space such that

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \mathrm{V}_h \tag{7.23}$$





Size of $\mathrm{V}_h \Rightarrow$ # of independent DOFs (here it's 12).

Note:

$$\underbrace{a(\boldsymbol{w}, \boldsymbol{w})}_{\text{2x (strain energy when imposing } \boldsymbol{w})} > 0 \text{ for } \boldsymbol{w} \in \mathrm{V} \quad (\boldsymbol{w} \neq \boldsymbol{0})$$

Also,

$$a(\boldsymbol{w}_h, \boldsymbol{w}_h) > 0 \text{ for } \boldsymbol{w}_h \in \mathrm{V}_h \quad (\mathrm{V}_h \subset \mathrm{V},\ w_h \neq 0)$$

Property I Define: $\boldsymbol{e}_h = \boldsymbol{u} - \boldsymbol{u}_h$.

$$\text{From (7.22), } a(\boldsymbol{u}, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \tag{7.24}$$

$$\text{From (7.23), } a(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \tag{7.25}$$

Hence,

$$a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}_h) = 0 \tag{7.26}$$

$$a(\boldsymbol{e}_h, \boldsymbol{v}_h) = 0 \tag{7.27}$$

(error is orthogonal in that sense to all $\boldsymbol{v}_h$ in F.E. space).

Property II

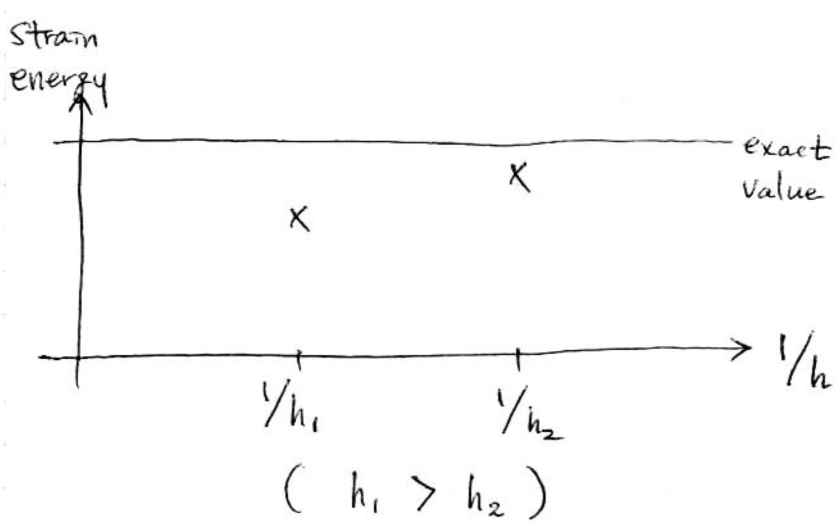
$$\boxed{a(\boldsymbol{u}_h, \boldsymbol{u}_h) \leq a(\boldsymbol{u}, \boldsymbol{u})} \tag{7.28}$$

Proof:

$$a(\boldsymbol{u}, \boldsymbol{u}) = a(\boldsymbol{u}_h + \boldsymbol{e}_h, \boldsymbol{u}_h + \boldsymbol{e}_h) \tag{7.29}$$

$$= a(\boldsymbol{u}_h, \boldsymbol{u}_h) + \underbrace{\cancel{2a(\boldsymbol{u}_h, \boldsymbol{e}_h)}}_{0 \text{ by Prop. I}} + \underbrace{a(\boldsymbol{e}_h, \boldsymbol{e}_h)}_{\geq 0} \tag{7.30}$$

$$\therefore a(\boldsymbol{u}, \boldsymbol{u}) \geq a(\boldsymbol{u}_h, \boldsymbol{u}_h) \tag{7.31}$$



2.094 — Finite Element Analysis of Solids and Fluids	Fall '08

Lecture 8 - Convergence of displacement-based FEM

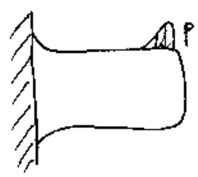
Prof. K.J. Bathe — MIT OpenCourseWare

(A) Find

$$\boldsymbol{u} \in \mathrm{V} \text{ such that } a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathrm{V} \text{ (Mathematical model)} \tag{8.1}$$

$$a(\boldsymbol{v}, \boldsymbol{v}) > 0 \quad \forall \boldsymbol{v} \in \mathrm{V}, \quad \boldsymbol{v} \neq \boldsymbol{0}. \tag{8.2}$$

where (8.2) implies that structures are supported properly. E.g.



(B) F.E. Problem Find

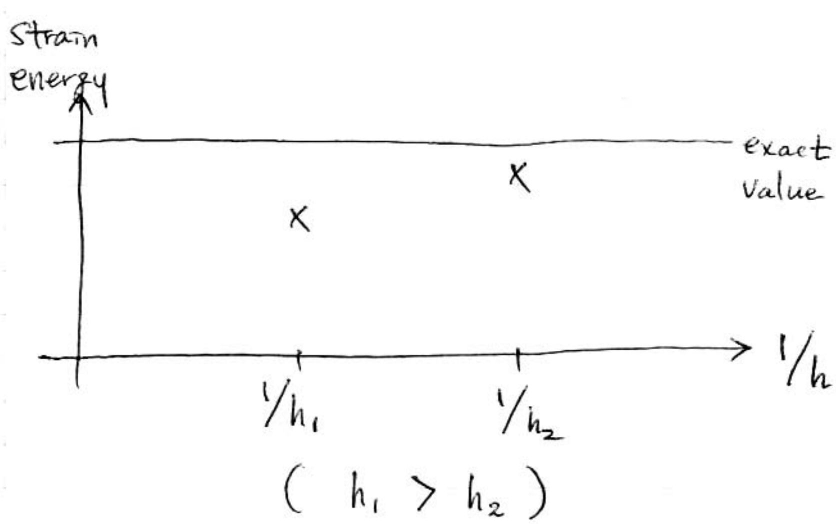
$$\boldsymbol{u}_h \in \mathrm{V}_h \text{ such that } a(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \mathrm{V}_h \tag{8.3}$$

$$a(\boldsymbol{v}_h, \boldsymbol{v}_h) > 0 \quad \forall \boldsymbol{v}_h \in \mathrm{V}_h, \quad \boldsymbol{v}_h \neq 0 \tag{8.4}$$

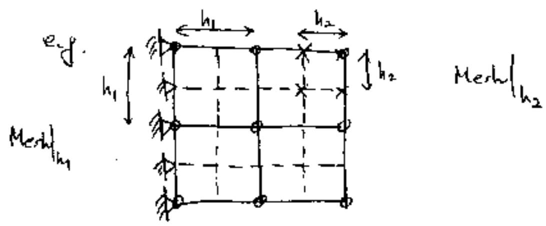
Properties $\boldsymbol{e}_h = \boldsymbol{u} - \boldsymbol{u}_h$

(I) $a(\boldsymbol{e}_h, \boldsymbol{v}_h) = 0 \quad \forall \boldsymbol{v}_h \in \mathrm{V}_h$ (8.5)

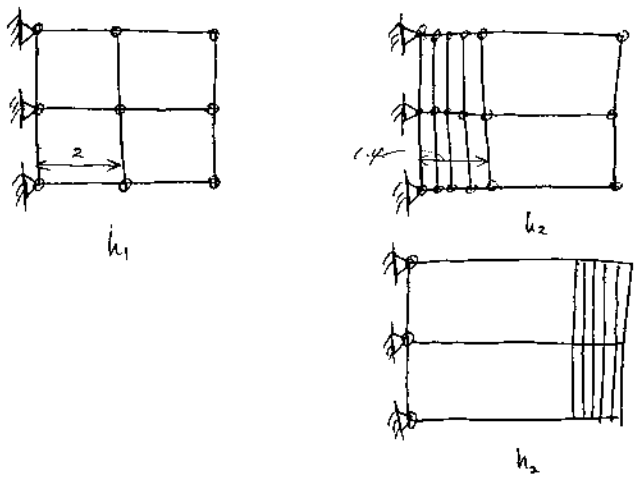
(II) $a(\boldsymbol{u}_h, \boldsymbol{u}_h) \leq a(\boldsymbol{u}, \boldsymbol{u})$ (8.6)



(C) Assume $\text{Mesh}\Big|_{h_1}$ "is contained in" $\text{Mesh}\Big|_{h_2}$



e.g. $\text{Mesh}\Big|_{h_1}$ not contained in $\text{Mesh}\Big|_{h_2}$



We assume (C), but need another property (independent of (C))

$$\textbf{(III)}\ a(\boldsymbol{e}_h, \boldsymbol{e}_h) \leq a(\boldsymbol{u}-\boldsymbol{v}_h, \boldsymbol{u}-\boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \mathrm{V}_h \tag{8.7}$$

$\boldsymbol{u}_h$ minimizes! (Recall $\boldsymbol{e}_h = \boldsymbol{u} - \boldsymbol{u}_h$)

Proof: Pick $\boldsymbol{w}_h \in \mathrm{V}_h$.

$$a(\boldsymbol{e}_h + \boldsymbol{w}_h, \boldsymbol{e}_h + \boldsymbol{w}_h) = a(\boldsymbol{e}_h, \boldsymbol{e}_h) + \cancel{2a(\boldsymbol{e}_h, \boldsymbol{w}_h)}^{\,0} + \underbrace{a(\boldsymbol{w}_h, \boldsymbol{w}_h)}_{\geq 0} \tag{8.8}$$

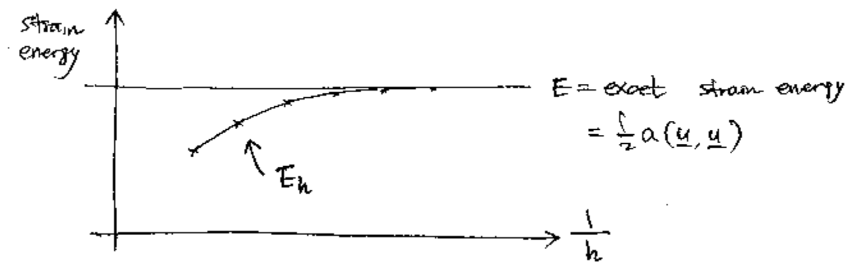
Equality holds for ($\boldsymbol{w}_h = \boldsymbol{0}$)

$$a(\boldsymbol{e}_h, \boldsymbol{e}_h) \leq a(\boldsymbol{e}_h + \boldsymbol{w}_h, \boldsymbol{e}_h + \boldsymbol{w}_h) \tag{8.9}$$
$$= a(\boldsymbol{u} - \boldsymbol{u}_h + \boldsymbol{w}_h, \boldsymbol{u} - \boldsymbol{u}_h + \boldsymbol{w}_h) \tag{8.10}$$

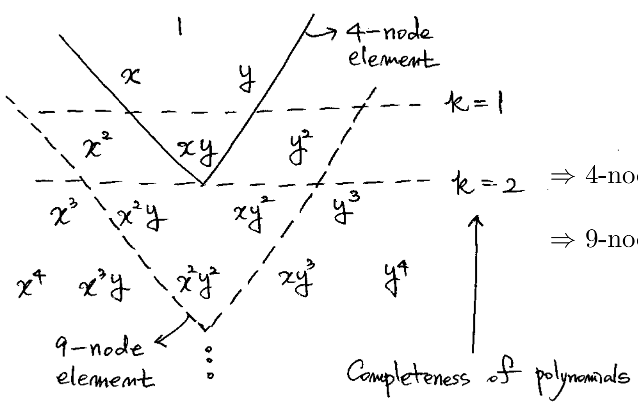
Take $\boldsymbol{w}_h = \boldsymbol{u}_h - \boldsymbol{v}_h$.

$$a(\boldsymbol{e}_h, \boldsymbol{e}_h) \leq a(\boldsymbol{u} - \boldsymbol{v}_h, \boldsymbol{u} - \boldsymbol{v}_h) \tag{8.11}$$

Using property (III) and (C), we can say that we will converge monotonically, from below, to $a(\boldsymbol{u}, \boldsymbol{u})$:



Pascal triangle (2D)



$\Rightarrow$ 4-node element is complete to $k = 1$.

$\Rightarrow$ 9-node element is complete to $k = 2$.

(Ch. 4.3)

$$\text{error in displacement} \sim C \cdot h^{k+1} \tag{8.12}$$

(C is a constant determined by the exact solution, material property...)

$$\text{error in stresses} \sim C \cdot h^{k} \tag{8.13}$$

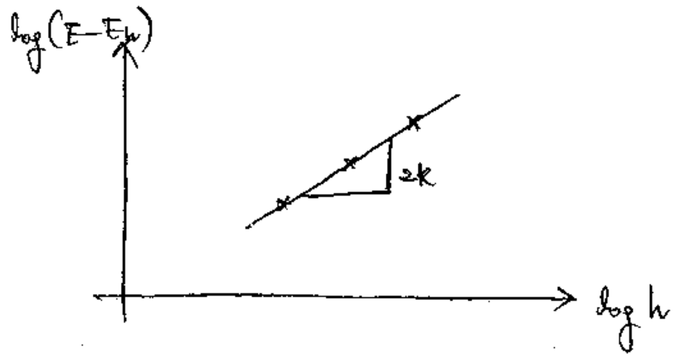
$$\boxed{\text{error in strain energy} \sim C \cdot h^{2k}} \qquad (\leftarrow \text{these } C \text{ are different}) \tag{8.14}$$

Hence,

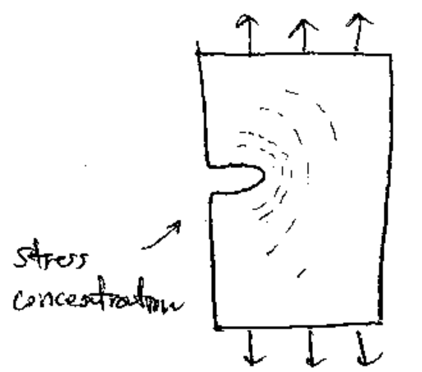
$$E - E_h = C \cdot h^{2k} \qquad \text{(roughly equal to)} \tag{8.15}$$

By theory,

$$\log(E - E_h) = \log C + 2k \log h \tag{8.16}$$



By experiment, we can evaluate $\log(E - E_h)$ for different meshes and plot $\log(E - E_h)$ vs. $\log h$



We need to use graded meshes if we have high stress gradients.

Example Consider an almost incompressible material:

$$\epsilon_V = \text{ vol. strain} \tag{8.17}$$

or

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} \rightarrow \text{ very small or zero} \tag{8.18}$$

We can "see" difficulties:

$$p = -\kappa\epsilon_V \quad \kappa = \text{ bulk modulus} \tag{8.19}$$

As the material becomes incompressible ($\nu = 0.3 \rightarrow 0.4999$)

$$\left.\begin{aligned} \kappa &\rightarrow \infty \\ \epsilon_V &\rightarrow 0 \end{aligned}\right\} \quad p \rightarrow \text{ finite number} \tag{8.20}$$

(Small error in ϵ_V results in huge error on pressure as $\kappa \rightarrow \infty$, the constant C in (8.15) can be very large $\Rightarrow$ locking)

2.094 — Finite Element Analysis of Solids and Fluids	Fall '08
Lecture 9 - u/p formulation	
Prof. K.J. Bathe	MIT OpenCourseWare

We want to solve

Reading: Sec. 4.4.3

I. Equilibrium

$$\begin{cases} \tau_{ij,j} + f_i^B = 0 & \text{in Volume} \\ \tau_{ij} n_j = f_i^{S_f} & \text{on } S_f \end{cases} \tag{9.1}$$

II. Compatibility

III. Stress-strain law

Use the principle of virtual displacements

$$\int_V \overline{\boldsymbol{\epsilon}}^T \boldsymbol{C} \boldsymbol{\epsilon} \, dV = \mathcal{R} \tag{9.2}$$

We recognize that if $\nu \to 0.5$

$$\epsilon_V \to 0 \quad (\epsilon_V = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) \tag{9.3}$$

$$\kappa = \frac{E}{3(1-2\nu)} \to \infty \tag{9.4}$$

$$p = -\kappa \epsilon_V \quad \text{must be accurately computed} \tag{9.5}$$

Solution

$$\tau_{ij} = \kappa \epsilon_V \delta_{ij} + 2G\epsilon'_{ij} \tag{9.6}$$

where

$$\delta_{ij} = \text{ Kronecker delta } = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{9.7}$$

Deviatoric strains:

$$\epsilon'_{ij} = \epsilon_{ij} - \frac{\epsilon_V}{3}\delta_{ij} \tag{9.8}$$

$$\tau_{ij} = -p\delta_{ij} + 2G\epsilon'_{ij} \quad \left(p = -\frac{\tau_{kk}}{3}\right) \tag{9.9}$$

(9.2) becomes

$$\int_V \overline{\boldsymbol{\epsilon}}'^T \boldsymbol{C}' \boldsymbol{\epsilon}' \, dV + \int_V \overline{\epsilon}_V \kappa \epsilon_V \, dV = \mathcal{R} \tag{9.10}$$

$$\int_V \overline{\boldsymbol{\epsilon}}'^T \boldsymbol{C}' \boldsymbol{\epsilon}' \, dV - \int_V \overline{\epsilon}_V^T p \, dV = \mathcal{R} \tag{9.11}$$

We need another equation because we now have another unknown p.

$$p + \kappa\epsilon_V = 0 \tag{9.12}$$

$$\int_V \overline{p}\left(p + \kappa\epsilon_V\right)\, dV = 0 \tag{9.13}$$

$$-\int_V \overline{p}\left(\epsilon_V + \frac{p}{\kappa}\right)\, dV = 0 \tag{9.14}$$

For an element,

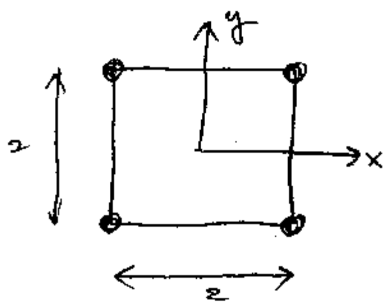
$$\boldsymbol{u} = \boldsymbol{H}\hat{\boldsymbol{u}} \tag{9.15}$$

$$\boldsymbol{\epsilon}' = \boldsymbol{B}_D\hat{\boldsymbol{u}} \tag{9.16}$$

$$\epsilon_V = \boldsymbol{B}_V\hat{\boldsymbol{u}} \tag{9.17}$$

$$p = \boldsymbol{H}_p\hat{\boldsymbol{p}} \tag{9.18}$$

Example



Plane strain ($\epsilon_{zz} = 0$)
4/1 element

Reading: Ex. 4.32 in the text

$$\epsilon_V = \epsilon_{xx} + \epsilon_{yy} \tag{9.19}$$

$$\boldsymbol{\epsilon}' = \begin{bmatrix} \epsilon_{xx} - \frac{1}{3}\left(\epsilon_{xx} + \epsilon_{yy}\right) \\ \epsilon_{yy} - \frac{1}{3}\left(\epsilon_{xx} + \epsilon_{yy}\right) \\ \gamma_{xy} \\ -\frac{1}{3}\left(\epsilon_{xx} + \epsilon_{yy}\right) \end{bmatrix} \tag{9.20}$$

Note: $\epsilon_{zz} = 0$ but $\epsilon'_{zz} \neq 0$!

$$p = \boldsymbol{H}_p\hat{\boldsymbol{p}} = [1]\{p_0\} \tag{9.21}$$

$$p(x, y) = p_0 \tag{9.22}$$

We obtain from (9.11) and (9.14)

$$\begin{bmatrix} \boldsymbol{K}_{uu} & \boldsymbol{K}_{up} \\ \boldsymbol{K}_{pu} & \boldsymbol{K}_{pp} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{u}} \\ \hat{\boldsymbol{p}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} \\ \boldsymbol{0} \end{bmatrix} \tag{9.23}$$

$$\boldsymbol{K}_{uu} = \int_V \boldsymbol{B}_D^T\boldsymbol{C}'\boldsymbol{B}_D\, dV \tag{9.24a}$$

$$\boldsymbol{K}_{up} = -\int_V \boldsymbol{B}_V^T\boldsymbol{H}_p\, dV \tag{9.24b}$$

$$\boldsymbol{K}_{pu} = -\int_V \boldsymbol{H}_p^T\boldsymbol{B}_V\, dV \tag{9.24c}$$

$$\boldsymbol{K}_{pp} = -\int_V \boldsymbol{H}_p^T\frac{1}{\kappa}\boldsymbol{H}_p\, dV \tag{9.24d}$$

In practice, we use elements that use pressure interpolations per element, not continuous between elements. For example:

9/1 el.

Then, unless $\nu = 0.5$ (where $\boldsymbol{K}_{pp} = \boldsymbol{0}$), we can use static condensation on the pressure dof's.

Use $\hat{\boldsymbol{p}}$ equations to eliminate $\hat{\boldsymbol{p}}$ from the $\hat{\boldsymbol{u}}$ equations.

$$\left(\boldsymbol{K}_{uu} - \boldsymbol{K}_{up}\boldsymbol{K}_{pp}^{-1}\boldsymbol{K}_{pu}\right)\hat{\boldsymbol{u}} = \boldsymbol{R} \tag{9.25}$$

(In practice, ν can be 0.499999...)

The "best element" is the 9/3 element. (9 nodes for displacement and 3 pressure dof's).

$$p(x, y) = p_0 + p_1 x + p_2 y \tag{9.26}$$

The inf-sup condition

Reading: Sec. 4.5

$$\underbrace{\inf}_{q_h \in \mathrm{Q}_h} \underbrace{\sup}_{v_h \in \mathrm{V}_h} \left[\frac{\int_{\mathrm{Vol}} q_h \overbrace{\boldsymbol{\nabla} \cdot \boldsymbol{v}_h}^{=\epsilon_V} \, d\mathrm{Vol}}{\underbrace{\|q_h\| \, \|\boldsymbol{v}_h\|}_{\text{for normalization}}} \right] \geq \beta > 0 \tag{9.27}$$

Q_h: pressure space.

If "this" holds, the element is optimal for the displacement assumption used (ellipticity must also be satisfied).

Note:

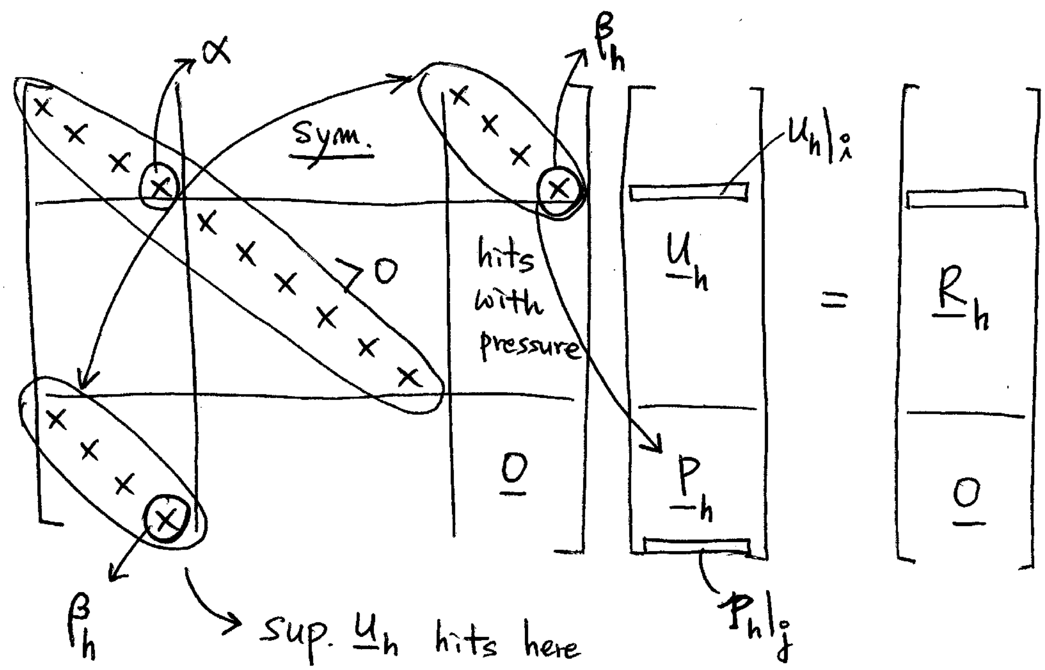
$$\begin{aligned} \text{infimum} &= \text{largest lower bound} \\ \text{supremum} &= \text{least upper bound} \end{aligned}$$

For example,

$$\begin{aligned} \inf \{1, 2, 4\} &= 1 \\ \sup \{1, 2, 4\} &= 4 \\ \inf \{x \in \mathrm{R}; \quad 0 < x < 2\} &= 0 \\ \sup \{x \in \mathrm{R}; \quad 0 < x < 2\} &= 2 \end{aligned}$$

(9.23) **rewritten** ($\kappa = \infty$, full incompressibility). Diagonalize using eigenvalues/eigenvectors.

For a mesh of element size h we want $\boxed{\beta_h > 0}$ as we refine the mesh, $h \to 0$



For  (entry [3,1] in matrix) assume the circled entry is the minimum (inf) of  .

Also, all entries in the matrix not shown are zero.

Case 1 $\beta_h = 0$

$$\Rightarrow \begin{cases} 0 \cdot u_h|_i = 0 & \text{(from the bottom equation)} \\ \underbrace{\alpha}_{\neq 0} \cdot u_h|_i + 0 \cdot p_h|_j = R_h|_i & \text{(from the top equation)} \end{cases}$$

$\Rightarrow$ no equation for $p_h|_j$

$\Rightarrow$ spurious pressure! (any pressure satisfies equation)

Case 2 $\beta_h = \text{small} = \epsilon$

$$\epsilon \cdot u_h|_i = 0 \quad \Rightarrow \boxed{u_h|_i = 0}$$

$$\therefore \epsilon \cdot \quad p_h|_j + u_h|_i \cdot \alpha = R_h|_i$$

$$\Rightarrow p_h|_j = \frac{R_h|_i}{\epsilon} \quad \Rightarrow \begin{pmatrix} \text{displ.} & = & 0 \\ \text{pressure} & \rightarrow & \text{large} \end{pmatrix} \text{ as } \epsilon \text{ is small}$$

The behavior of given mesh when bulk modulus increases: locking, large pressures. See Example 4.39 textbook.

2.094 — Finite Element Analysis of Solids and Fluids Fall ‘08

Lecture 10 - F.E. large deformation/general nonlinear analysis

Prof. K.J. Bathe MIT OpenCourseWare

We developed

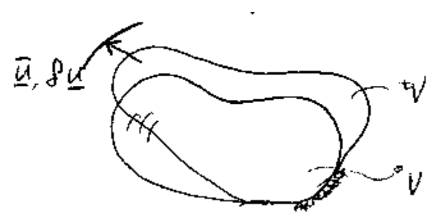
Reading: Ch. 6

$$\int_{^tV} {}^t\tau_{ij}\, {}_t\bar{e}_{ij}\, d^tV = {}^t\mathcal{R} \tag{10.1}$$

$${}_t\bar{e}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial {}^tx_j} + \frac{\partial \overline{u}_j}{\partial {}^tx_i}\right) \tag{10.2}$$

$$\int_{^tV} {}^t\tau_{ij}\, \delta_t e_{ij}\, d^tV = {}^t\mathcal{R} \tag{10.3}$$

$$\delta_t e_{ij} = \frac{1}{2}\left(\frac{\partial(\delta u_i)}{\partial\, {}^tx_j} + \frac{\partial(\delta u_j)}{\partial\, {}^tx_i}\right) \quad (\equiv\ {}_t\bar{e}_{ij}) \tag{10.4}$$



In FEA:

$${}^t\boldsymbol{F} = {}^t\boldsymbol{R} \tag{10.5}$$

In linear analysis

$${}^t\boldsymbol{F} = \boldsymbol{K}\, {}^t\boldsymbol{U} \Rightarrow \boldsymbol{K}\boldsymbol{U} = \boldsymbol{R} \tag{10.6}$$

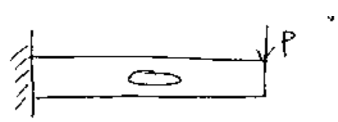
In general nonlinear analysis, we need to iterate. Assume the solution is known “at time t”

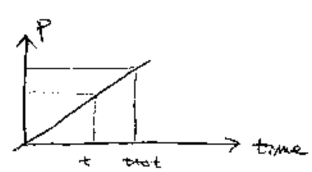
$${}^t\boldsymbol{x} = {}^0\boldsymbol{x} + {}^t\boldsymbol{u} \tag{10.7}$$

Hence ${}^t\boldsymbol{F}$ is known. Then we consider

$${}^{t+\Delta t}\boldsymbol{F} = {}^{t+\Delta t}\boldsymbol{R} \tag{10.8}$$

Consider the loads (applied external loads) to be deformation-independent, e.g.





Then we can write

$$^{t+\Delta t}\boldsymbol{F} = {}^{t}\boldsymbol{F} + \boldsymbol{F} \tag{10.9}$$

$$^{t+\Delta t}\boldsymbol{U} = {}^{t}\boldsymbol{U} + \boldsymbol{U} \tag{10.10}$$

where only ${}^{t}\boldsymbol{F}$ and ${}^{t}\boldsymbol{U}$ are known.

$$\boldsymbol{F} \cong {}^{t}\boldsymbol{K}\Delta\boldsymbol{U}, \quad {}^{t}\boldsymbol{K} = \text{tangent stiffness matrix at time } t \tag{10.11}$$

From (10.8),

$$^{t}\boldsymbol{K}\Delta\boldsymbol{U} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t}\boldsymbol{F} \tag{10.12}$$

We use this to obtain an approximation to $\boldsymbol{U}$. We obtain a more accurate solution for $\boldsymbol{U}$ (i.e. ${}^{t+\Delta t}\boldsymbol{F}$) using

$$^{t+\Delta t}\boldsymbol{K}^{(i-1)}\Delta\boldsymbol{U}^{(i)} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F}^{(i-1)} \tag{10.13}$$

$$^{t+\Delta t}\boldsymbol{U}^{(i)} = {}^{t+\Delta t}\boldsymbol{U}^{(i-1)} + \Delta\boldsymbol{U}^{(i)} \tag{10.14}$$

Also,

$$^{t+\Delta t}\boldsymbol{F}^{(0)} = {}^{t}\boldsymbol{F} \tag{10.15}$$

$$^{t+\Delta t}\boldsymbol{K}^{(0)} = {}^{t}\boldsymbol{K} \tag{10.16}$$

$$^{t+\Delta t}\boldsymbol{U}^{(0)} = {}^{t}\boldsymbol{U} \tag{10.17}$$

Iterate for $i = 1, 2, 3\ldots$ until convergence. Convergence is reached when

$$\left\|\Delta\boldsymbol{U}^{(i)}\right\|_2 < \epsilon_D \tag{10.18}$$

$$\left\|{}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F}^{(i-1)}\right\|_2 < \epsilon_F \tag{10.19}$$

Note:

$$\|\boldsymbol{a}\|_2 = \sqrt{\sum_i (a_i)^2}$$

$$\sum_{i=1,2,3\ldots} \Delta\boldsymbol{U}^{(i)} = \boldsymbol{U}$$

$\Delta\boldsymbol{U}^{(1)}$ in (10.13) is $\Delta\boldsymbol{U}$ in (10.12).

(10.13) is the *full* Newton-Raphson iteration.

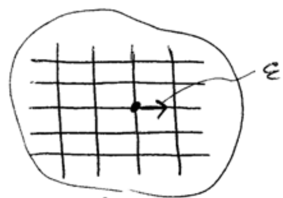
How we could (in principle) calculate ${}^{t}\boldsymbol{K}$

Process

- Increase the displacement ${}^{t}U_i$ by ϵ, with no increment for all ${}^{t}U_j$, $j \neq i$
- calculate ${}^{t+\epsilon}\boldsymbol{F}$
- the i-th column in ${}^{t}\boldsymbol{K} = \left({}^{t+\epsilon}\boldsymbol{F} - {}^{t}\boldsymbol{F}\right)/\epsilon = \frac{\partial\,{}^{t}\boldsymbol{F}}{\partial\,{}^{t}U_i}$.

So, perform this process for $i = 1, 2, 3, \ldots, n$, where n is the total number of degrees of freedom. Pictorially,

$$
{}^t\boldsymbol{K} = \begin{bmatrix} \vdots & \vdots & \\ \vdots & \vdots & \cdots \\ \vdots & \vdots & \end{bmatrix}
$$



in deformed geometry.

A **general** difficulty: we cannot "simply" increment Cauchy stresses.

- ${}^{t+\Delta t}\tau_{ij}$ referred to area at time $t + \Delta t$
- ${}^t\tau_{ij}$ referred to area at time t.

We define a new stress measure, *2nd Piola - Kirchhoff stress*, ${}^{t+\Delta t}_{\ \ 0}S_{ij}$, where 0 in the leading subscript refers to original configuration. Then,

$$
{}^{t+\Delta t}_{\ \ 0}S_{ij} = {}^t_0S_{ij} + {}_0S_{ij} \tag{10.20}
$$

The strain measure energy-conjugate to the 2nd P-K stress ${}^t_0S_{ij}$ is the Green-Lagrange strain ${}^t_0\epsilon_{ij}$

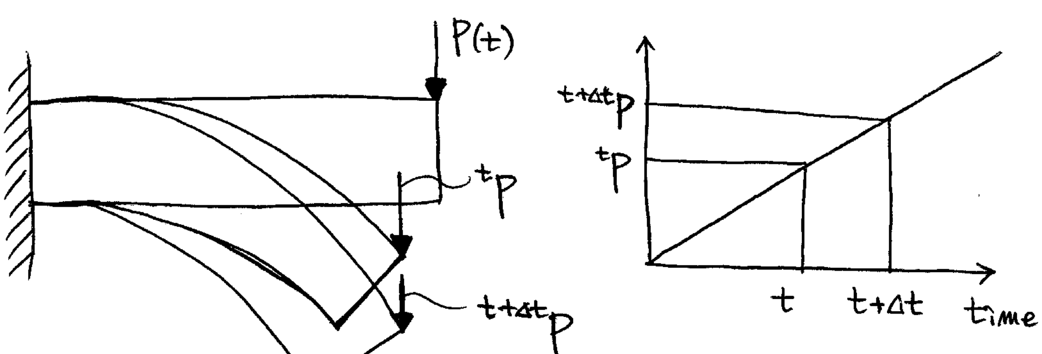
Then,

$$
\int_{{}^0V} {}^t_0S_{ij}\ \delta\, {}^t_0\epsilon_{ij}\ d^0V = {}^t\mathcal{R} \tag{10.21}
$$

Also,

$$
\int_{{}^0V} {}^{t+\Delta t}_{\ \ 0}S_{ij}\ \delta\, {}^{t+\Delta t}_{\ \ 0}\epsilon_{ij}\ d^0V = {}^{t+\Delta t}\mathcal{R} \tag{10.22}
$$

Example



$$
{}^t\underline{F} = {}^t\underline{R}
$$

$$ {}^{t}\boldsymbol{F} = {}^{t}\boldsymbol{R} \tag{10.23} $$
$$ {}^{t+\Delta t}\boldsymbol{F} = {}^{t+\Delta t}\boldsymbol{R} \tag{10.24} $$

(every time it is in equilibrium)

(10.13) and (10.14) give:

$i = 1$,

$$ {}^{t+\Delta t}\boldsymbol{K}^{(0)}\ \Delta\boldsymbol{U}^{(1)} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F}^{(0)} \equiv \mathrm{fn}({}^{t}\boldsymbol{U}) \tag{10.25} $$
$$ {}^{t+\Delta t}\boldsymbol{U}^{(1)} = {}^{t+\Delta t}\boldsymbol{U}^{(0)} + \Delta\boldsymbol{U}^{(1)} \tag{10.26} $$

$i = 2$,

$$ {}^{t+\Delta t}\boldsymbol{K}^{(1)}\ \Delta\boldsymbol{U}^{(2)} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F}^{(1)} \tag{10.27} $$
$$ {}^{t+\Delta t}\boldsymbol{U}^{(2)} = {}^{t+\Delta t}\boldsymbol{U}^{(1)} + \Delta\boldsymbol{U}^{(2)} \tag{10.28} $$

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Lecture 11 - Deformation, strain and stress tensors

Prof. K.J. Bathe MIT OpenCourseWare

We stated that we use

Reading: Ch. 6

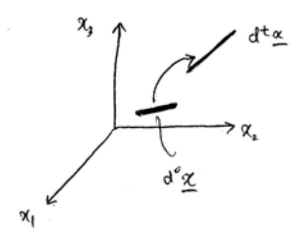
$$\int_{{}^tV} {}^t\tau_{ij}\ \delta_t e_{ij}\ d\,{}^tV = \int_{{}^0V} {}^t_0S_{ij}\ \delta\,{}^t_0\epsilon_{ij}\ d\,{}^0V = {}^t\mathcal{R} \tag{11.1}$$

The deformation gradient We use ${}^tx_i = {}^0x_i + {}^tu_i$

$$ {}^t_0\boldsymbol{X} = \begin{bmatrix} \frac{\partial\,{}^tx_1}{\partial\,{}^0x_1} & \frac{\partial\,{}^tx_1}{\partial\,{}^0x_2} & \frac{\partial\,{}^tx_1}{\partial\,{}^0x_3} \\ \frac{\partial\,{}^tx_2}{\partial\,{}^0x_1} & \frac{\partial\,{}^tx_2}{\partial\,{}^0x_2} & \frac{\partial\,{}^tx_2}{\partial\,{}^0x_3} \\ \frac{\partial\,{}^tx_3}{\partial\,{}^0x_1} & \frac{\partial\,{}^tx_3}{\partial\,{}^0x_2} & \frac{\partial\,{}^tx_3}{\partial\,{}^0x_3} \end{bmatrix} \tag{11.2}$$

$$d\,{}^t\boldsymbol{x} = \begin{bmatrix} d\,{}^tx_1 \\ d\,{}^tx_2 \\ d\,{}^tx_3 \end{bmatrix} \tag{11.3}$$

$$d\,{}^0\boldsymbol{x} = \begin{bmatrix} d\,{}^0x_1 \\ d\,{}^0x_2 \\ d\,{}^0x_3 \end{bmatrix} \tag{11.4}$$



Implies that

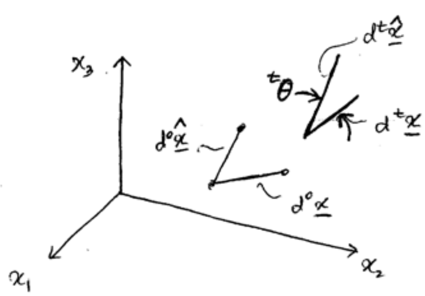
$$\boxed{d\,{}^t\boldsymbol{x} = {}^t_0\boldsymbol{X}\,d\,{}^0\boldsymbol{x}} \tag{11.5}$$

(${}^t_0\boldsymbol{X}$ is frequently denoted by ${}^t_0\boldsymbol{F}$ or simply $\boldsymbol{F}$, but we use $\boldsymbol{F}$ for force vector)

We will also use the right Cauchy-Green deformation tensor

$$\boxed{{}^t_0\boldsymbol{C} = {}^t_0\boldsymbol{X}^T\,{}^t_0\boldsymbol{X}} \tag{11.6}$$

Some applications



The stretch of a fiber (${}^t\lambda$):

$$\left({}^t\lambda\right)^2 = \frac{d\,{}^t\boldsymbol{x}^T d\,{}^t\boldsymbol{x}}{d\,{}^0\boldsymbol{x}^T d\,{}^0\boldsymbol{x}} = \left(\frac{d\,{}^ts}{d\,{}^0s}\right)^2 \tag{11.7}$$

The length of a fiber is

$$d\,{}^0s = \left(d\,{}^0\boldsymbol{x}^T d\,{}^0\boldsymbol{x}\right)^{\frac{1}{2}} \tag{11.8}$$

$$\left({}^t\lambda\right)^2 = \frac{\left(d\,{}^0\boldsymbol{x}^T\,{}^t_0\boldsymbol{X}^T\right)\left({}^t_0\boldsymbol{X}\,d\,{}^0\boldsymbol{x}\right)}{d\,{}^0s\cdot d\,{}^0s}, \qquad \text{from (11.5)} \tag{11.9}$$

Express

$$d\,{}^0\boldsymbol{x} = \left(d\,{}^0s\right)\,{}^0\boldsymbol{n} \tag{11.10}$$

$${}^0\boldsymbol{n} = \text{ unit vector into direction of } d\,{}^0\boldsymbol{x} \tag{11.11}$$

$$\Rightarrow \left({}^t\lambda\right)^2 = {}^0\boldsymbol{n}^T\,{}^t_0\boldsymbol{C}\,{}^0\boldsymbol{n} \tag{11.12}$$

$$\therefore \boxed{{}^t\lambda = \left({}^0\boldsymbol{n}^T\,{}^t_0\boldsymbol{C}\,{}^0\boldsymbol{n}\right)^{\frac{1}{2}}} \tag{11.13}$$

Also,

$$\left(d\,{}^t\hat{\boldsymbol{x}}\right)^T\cdot\left(d\,{}^t\boldsymbol{x}\right) = \left(d\,{}^t\hat{s}\right)\left(d\,{}^ts\right)\cos{}^t\theta, \qquad \left(\boldsymbol{a}\cdot\boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\|\cos\theta\right) \tag{11.14}$$

From (11.5),

$$\cos{}^t\theta = \frac{\left(d\,{}^0\hat{\boldsymbol{x}}^T\,{}^t_0\hat{\boldsymbol{X}}^T\right)\left({}^t_0\boldsymbol{X}\,d\,{}^0\boldsymbol{x}\right)}{d\,{}^t\hat{s}\,d\,{}^ts} \qquad \left({}^t_0\hat{\boldsymbol{X}} \equiv {}^t_0\boldsymbol{X}\right) \tag{11.15}$$

$$= \frac{d\,{}^0\hat{s}\,{}^0\hat{\boldsymbol{n}}^T\,{}^t_0\boldsymbol{C}\,{}^0\boldsymbol{n}\,d\,{}^0s}{d\,{}^t\hat{s}\cdot d\,{}^ts} \tag{11.16}$$

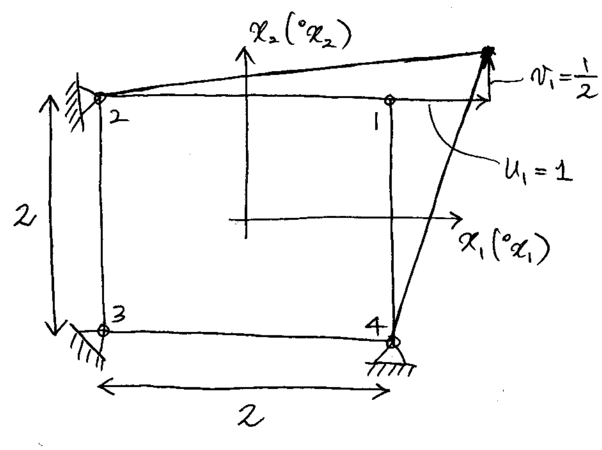
$$\therefore \boxed{\cos{}^t\theta = \frac{{}^0\hat{\boldsymbol{n}}^T\,{}^t_0\boldsymbol{C}\,{}^0\boldsymbol{n}}{{}^t\hat{\lambda}\,{}^t\lambda}} \tag{11.17}$$

Also,

$$\boxed{{}^t\rho = \frac{{}^0\rho}{\det{}^t_0\boldsymbol{X}}} \qquad \text{(see Ex. 6.5)} \tag{11.18}$$

Example

Reading: Ex. 6.6 in the text



$$h_1 = \frac{1}{4}(1 + {}^0x_1)(1 + {}^0x_2) \tag{11.19}$$

$$\vdots$$

$${}^tx_i = {}^0x_i + {}^tu_i \tag{11.20}$$

$$= \sum_{k=1}^{4} h_k\, {}^tx_i^k, \quad (i = 1, 2) \tag{11.21}$$

where ${}^tx_i^k$ are the nodal point coordinates at time t (${}^tx_1^1 = 2$, ${}^tx_2^1 = 1.5$)

Then we obtain

$${}_0^t\boldsymbol{X} = \frac{1}{4}\begin{bmatrix} 5 + {}^0x_2 & 1 + {}^0x_1 \\ \frac{1}{2}(1 + {}^0x_2) & \frac{1}{2}(9 + {}^0x_1) \end{bmatrix} \tag{11.22}$$

At ${}^0x_1 = 0$, ${}^0x_2 = 0$,

$${}_0^t\boldsymbol{X}\Big|_{{}^0x_i = {}^0x_2 = 0} = \frac{1}{4}\begin{bmatrix} 5 & 1 \\ \frac{1}{2} & \frac{9}{2} \end{bmatrix} \tag{11.23}$$

The Green-Lagrange Strain

$${}_0^t\epsilon = \frac{1}{2}\left({}_0^t\boldsymbol{X}^T\, {}_0^t\boldsymbol{X} - \boldsymbol{I}\right) = \frac{1}{2}\left({}_0^t\boldsymbol{C} - \boldsymbol{I}\right) \tag{11.24}$$

$$\frac{\partial\, {}^tx_i}{\partial\, {}^0x_j} = \frac{\partial\left({}^0x_i + {}^tu_i\right)}{\partial\, {}^0x_j} = \delta_{ij} + \frac{\partial\, {}^tu_i}{\partial\, {}^0x_j} \tag{11.25}$$

We find that

$${}_0^t\epsilon_{ij} = \frac{1}{2}\left({}_0^tu_{i,j} + {}_0^tu_{j,i} + {}_0^tu_{k,i}\, {}_0^tu_{k,j}\right), \qquad \text{sum over } k = 1, 2, 3 \tag{11.26}$$

where

$${}_0^tu_{i,j} = \frac{\partial\, {}^tu_i}{\partial\, {}^0x_j} \tag{11.27}$$

Polar decomposition of ${}_0^t\boldsymbol{X}$

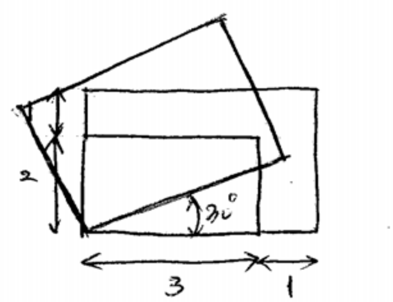
$$ {}_0^t\boldsymbol{X} = {}_0^t\boldsymbol{R}\,{}_0^t\boldsymbol{U} \tag{11.28} $$

where ${}_0^t\boldsymbol{R}$ is a rotation matrix, such that

$$ {}_0^t\boldsymbol{R}^T\,{}_0^t\boldsymbol{R} = \boldsymbol{I} \tag{11.29} $$

and ${}_0^t\boldsymbol{U}$ is a symmetric matrix (stretch)

Ex. 6.9 textbook



$$ {}_0^t\boldsymbol{X} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{4}{3} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \tag{11.30} $$

Then,

$$ {}_0^t\boldsymbol{C} = {}_0^t\boldsymbol{X}^T\,{}_0^t\boldsymbol{X} = \left({}_0^t\boldsymbol{U}\right)^2 \tag{11.31} $$

$$ {}_0^t\boldsymbol{\epsilon} = \frac{1}{2}\left[\left({}_0^t\boldsymbol{U}\right)^2 - \boldsymbol{I}\right] \tag{11.32} $$

This shows, by an example, that the components of the Green-Lagrange strain are independent of a rigid-body rotation.

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Lecture 12 - Total Lagrangian formulation

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We discussed:

$$ {}_0^t\boldsymbol{X} = \left[\frac{\partial\, {}^t x_i}{\partial\, {}^0 x_j}\right] \qquad \Rightarrow \quad d\,{}^t\boldsymbol{x} = {}_0^t\boldsymbol{X}\, d\,{}^0\boldsymbol{x}, \quad d\,{}^0\boldsymbol{x} = \left({}_0^t\boldsymbol{X}\right)^{-1} d\,{}^t\boldsymbol{x} \tag{12.1}$$

$$ {}_0^t\boldsymbol{C} = {}_0^t\boldsymbol{X}^T\, {}_0^t\boldsymbol{X} \tag{12.2}$$

$$ d\,{}^0\boldsymbol{x} = {}_t^0\boldsymbol{X}\, d\,{}^t\boldsymbol{x} \qquad \text{where } {}_t^0\boldsymbol{X} = \left({}_0^t\boldsymbol{X}\right)^{-1} = \left[\frac{\partial\, {}^0 x_i}{\partial\, {}^t x_j}\right] \tag{12.3}$$

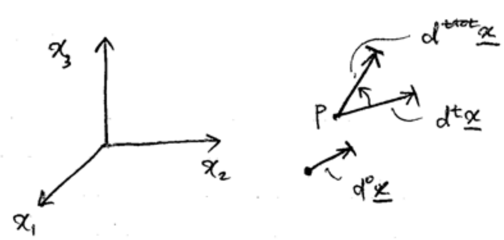
The Green-Lagrange strain:

$$ {}_0^t\boldsymbol{\epsilon} = \frac{1}{2}\left({}_0^t\boldsymbol{X}^T\, {}_0^t\boldsymbol{X} - \boldsymbol{I}\right) = \frac{1}{2}\left({}_0^t\boldsymbol{C} - \boldsymbol{I}\right) \tag{12.4}$$

Polar decomposition:

$$ {}_0^t\boldsymbol{X} = {}_0^t\boldsymbol{R}\,{}_0^t\boldsymbol{U} \Rightarrow {}_0^t\boldsymbol{\epsilon} = \frac{1}{2}\left(\left({}_0^t\boldsymbol{U}\right)^2 - \boldsymbol{I}\right) \tag{12.5}$$

We see, physically that:



where $d^{t+\Delta t}\boldsymbol{x}$ and $d^t\boldsymbol{x}$ are the same lengths $\Rightarrow$ the components of the G-L strain do not change.

Note in FEA

$$ \left.\begin{aligned} {}^0 x_i &= \sum_k h_k\, {}^0 x_i^k \\ {}^t u_i &= \sum_k h_k\, {}^t u_i^k \end{aligned}\right\} \quad \text{for an element} \tag{12.6}$$

$$ {}^t x_i = {}^0 x_i + {}^t u_i \rightarrow \quad \text{for any particle} \tag{12.7}$$

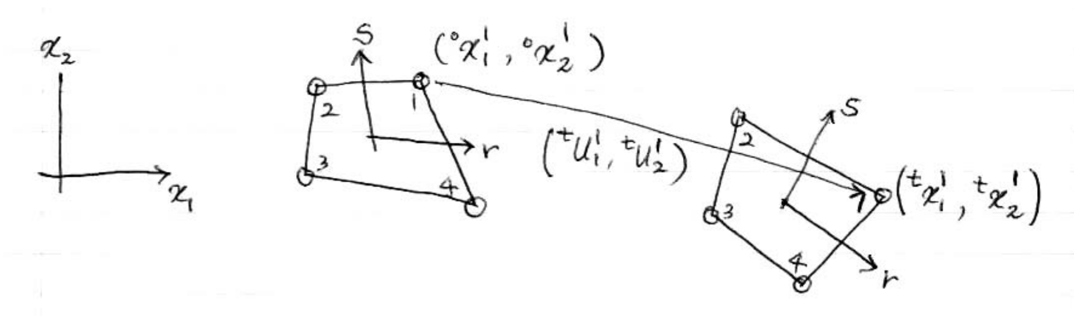
Hence for the element

$$ {}^t x_i = \sum_k h_k\, {}^0 x_i^k + \sum_k h_k\, {}^t u_i^k \tag{12.8}$$

$$ = \sum_k h_k \left({}^0 x_i^k + {}^t u_i^k\right) \tag{12.9}$$

$$ = \sum_k h_k\, {}^t x_k^k \tag{12.10}$$

E.g., $k = 4$



2nd Piola-Kirchhoff stress

$$ {}_0^t\boldsymbol{S} = \frac{{}^0\rho}{{}^t\rho}\,{}_t^0\boldsymbol{X}\,{}^t\boldsymbol{\tau}\,{}_t^0\boldsymbol{X}^T \rightarrow \quad \text{components also independent of a rigid body rotation} \tag{12.11} $$

Then

$$ \int_{{}^0V} {}_0^tS_{ij}\, \delta\, {}_0^t\epsilon_{ij}\, d^0V = \int_{{}^tV} {}^t\tau_{ij}\, \delta\, {}_te_{ij}\, d^tV = {}^t\mathcal{R} \tag{12.12} $$

We can use an incremental decomposition of stress/strain.

$$ {}_0^{t+\Delta t}\boldsymbol{S} = {}_0^t\boldsymbol{S} + {}_0\boldsymbol{S} \tag{12.13} $$
$$ {}_0^{t+\Delta t}S_{ij} = {}_0^tS_{ij} + {}_0S_{ij} \tag{12.14} $$
$$ {}_0^{t+\Delta t}\boldsymbol{\epsilon} = {}_0^t\boldsymbol{\epsilon} + {}_0\boldsymbol{\epsilon} \tag{12.15} $$
$$ {}_0^{t+\Delta t}\epsilon_{ij} = {}_0^t\epsilon_{ij} + {}_0\epsilon_{ij} \tag{12.16} $$

Assume the solution is kown at time t, calculate the solution at time $t+\Delta t$. Hence, we apply (12.12) at time $t+\Delta t$:

$$ \int_{{}^0V} {}_0^{t+\Delta t}S_{ij}\, \delta\, {}_0^{t+\Delta t}\epsilon_{ij}\, d^0V = {}^{t+\Delta t}\mathcal{R} \tag{12.17} $$

Look at $\delta\, {}_0^t\epsilon_{ij}$:

$$ \delta\, {}_0^t\epsilon_{ij} = \delta \frac{1}{2}\left({}_0^tu_{i,j} + {}_0^tu_{j,i} + {}_0^tu_{k,i}\,{}_0^tu_{k,j} \right) \tag{12.18a} $$
$$ \delta\, {}_0^t\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial {}^0x_j} + \frac{\partial \delta u_j}{\partial {}^0x_i} + \frac{\partial \delta u_k}{\partial {}^0x_i} \cdot \frac{\partial {}^tu_k}{\partial {}^0x_j} + \frac{\partial {}^tu_k}{\partial {}^0x_i} \cdot \frac{\partial \delta u_k}{\partial {}^0x_j} \right) \tag{12.18b} $$
$$ \delta\, {}_0^t\epsilon_{ij} = \frac{1}{2}\left(\delta\, {}_0u_{i,j} + \delta\, {}_0u_{j,i} + \delta\, {}_0u_{k,i}\,{}_0^tu_{k,j} + {}_0^tu_{k,i}\,\delta\, {}_0u_{k,j} \right) \tag{12.18c} $$

We have

$$ {}_0^{t+\Delta t}\epsilon_{ij} - {}_0^t\epsilon_{ij} = {}_0\epsilon_{ij} \tag{12.19} $$
$$ {}_0\epsilon_{ij} = {}_0e_{ij} + {}_0\eta_{ij} \tag{12.20} $$

where ${}_0e_{ij}$ is the linear incremental strain, ${}_0\eta_{ij}$ is the nonlinear incremental strain, and

$$ {}_0e_{ij} = \frac{1}{2}\left({}_0u_{i,j} + {}_0u_{j,i} + \underbrace{{}^t_0u_{k,i}\,{}_0u_{k,j} + {}_0u_{k,i}\,{}^t_0u_{k,j}}_{\text{initial displ. effect}}\right) \tag{12.21} $$

$$ {}_0\eta_{ij} = \frac{1}{2}\,{}_0u_{k,i}\,{}_0u_{k,j} \tag{12.22} $$

where

$$ {}_0u_{k,j} = \frac{\partial u_k}{\partial^0 x_j}, \qquad \boxed{u_k = {}^{t+\Delta t}u_k - {}^tu_k} \tag{12.23} $$

Note

$$ \delta\,{}^{t+\Delta t}\epsilon_{ij} = \delta\,{}_0\epsilon_{ij} \qquad \left(\because\ \delta\,{}^t_0\epsilon_{ij} = 0 \text{ when changing the configuration at } t+\Delta t\right) \tag{12.24} $$

From (12.17):

$$ \int_{{}^0V} \left({}^t_0S_{ij} + {}_0S_{ij}\right)\left(\delta_0e_{ij} + \delta_0\eta_{ij}\right) d^0V $$

$$ = \int_{{}^0V} \left({}^t_0S_{ij}\delta_0e_{ij} + {}_0S_{ij}\delta_0e_{ij} + {}^t_0S_{ij}\delta_0\eta_{ij} + {}_0S_{ij}\delta_0\eta_{ij}\right) d^0V \tag{12.25} $$

$$ = {}^{t+\Delta t}\mathcal{R} \tag{12.26} $$

Linearization

$$ \int_{{}^0V}\left(\overbrace{{}_0S_{ij}\delta_0e_{ij}}^{{}^t_0\boldsymbol{K}_L\boldsymbol{U}} + \overbrace{{}^t_0S_{ij}\delta_0\eta_{ij}}^{{}^t_0\boldsymbol{K}_{NL}\boldsymbol{U}}\right) d^0V = {}^{t+\Delta t}\mathcal{R} - \underbrace{\int_{{}^0V} {}^t_0S_{ij}\delta_0e_{ij}\,d^0V}_{{}^t_0\boldsymbol{F}} \tag{12.27} $$

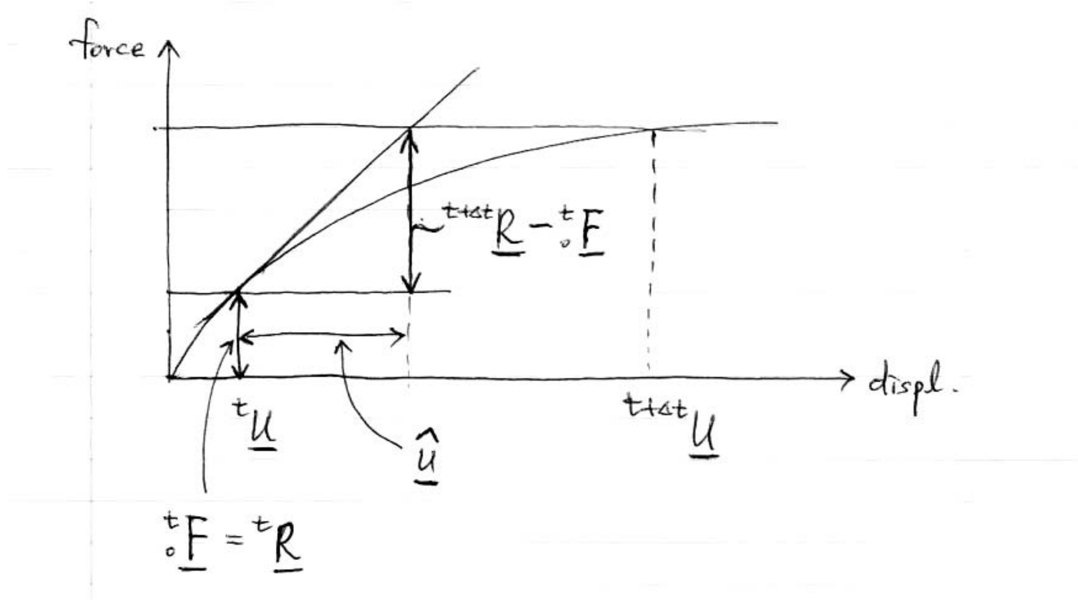
We use,

$$ {}_0S_{ij} \simeq {}_0C_{ijrs}\,{}_0e_{rs} \tag{12.28} $$

We arrive at, with the finite element interpolations,

$$ \left({}^t_0\boldsymbol{K}_L + {}^t_0\boldsymbol{K}_{NL}\right)\boldsymbol{U} = {}^{t+\Delta t}\boldsymbol{R} - {}^t_0\boldsymbol{F} \tag{12.29} $$

where $\boldsymbol{U}$ is the nodal displacement increment.



Left hand side as before but using $(k-1)$ and right hand side is

$$= {}^{t+\Delta t}\mathcal{R} - \int_{^0V} {}^{t+\Delta t}_{\ \ 0}S_{ij}\,\delta\,{}^{t+\Delta t}_{\ \ 0}\epsilon^{(k-1)}_{ij}\, d^0V \tag{12.30}$$

gives

$$ {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}_{\ \ 0}\boldsymbol{F}^{(k-1)} \tag{12.31}$$

In the full N-R iteration, we use

$$\left({}^{t+\Delta t}_{\ \ 0}\boldsymbol{K}^{(k-1)}_{L} + {}^{t+\Delta t}_{\ \ 0}\boldsymbol{K}^{(k-1)}_{NL}\right)\Delta\boldsymbol{U}^{(k)} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}_{\ \ 0}\boldsymbol{F}^{(k-1)} \tag{12.32}$$

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Lecture 13 - Total Lagrangian formulation, cont'd

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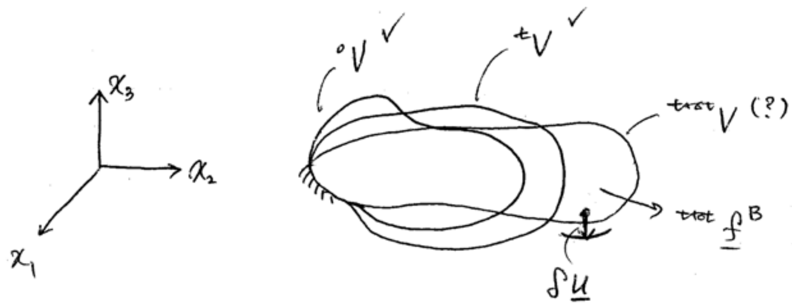
Example truss element. Recall:



Principle of virtual displacements applied at some time $t + \Delta t$:

$$\int_{^{t+\Delta t}V} {}^{t+\Delta t}\tau_{ij}\,\delta_{t+\Delta t}e_{ij}\,d^{t+\Delta t}V = {}^{t+\Delta t}\mathcal{R} \tag{13.1}$$

$$\int_{^0V} {}^{t+\Delta t}_{\ \ 0}S_{ij}\,\delta\,{}^{t+\Delta t}_{\ \ 0}\epsilon_{ij}\,\delta^0V = {}^{t+\Delta t}\mathcal{R} \tag{13.2}$$

$${}^{t+\Delta t}_{\ \ 0}S_{ij} = {}^t_0S_{ij} + {}_0S_{ij} \tag{13.3}$$

$${}^{t+\Delta t}_{\ \ 0}\epsilon_{ij} = {}^t_0\epsilon_{ij} + {}_0\epsilon_{ij} \tag{13.4}$$

$${}_0\epsilon_{ij} = {}_0e_{ij} + {}_0\eta_{ij} \tag{13.5}$$

where ${}^t_0S_{ij}$ and ${}^t_0\epsilon_{ij}$ are known, but ${}_0S_{ij}$ and ${}_0\epsilon_{ij}$ are not.

$${}_0e_{ij} = \frac{1}{2}\left({}_0u_{i,j} + {}_0u_{j,i} + {}^t_0u_{k,i}\,{}_0u_{k,j} + {}^t_0u_{k,j}\,{}_0u_{k,i}\right) \tag{13.6}$$

$${}_0\eta_{ij} = \frac{1}{2}\left({}_0u_{k,i}\,{}_0u_{k,j}\right) \tag{13.7}$$

Substitute into (13.2) and linearize to obtain

$$\int_{^0V} \delta_0e_{ij}\,{}_0C_{ijrs}\,{}_0e_{rs}\,d^0V + \int_{^0V} {}^t_0S_{ij}\,\delta_0\eta_{ij}\,d^0V = {}^{t+\Delta t}\mathcal{R} - \int_{^0V} \delta_0e_{ij}\,{}^t_0S_{ij}\,d^0V \tag{13.8}$$

F.E. discretization gives

$$\left({}^t_0\boldsymbol{K}_L + {}^t_0\boldsymbol{K}_{NL}\right)\Delta\boldsymbol{U} = {}^{t+\Delta t}\boldsymbol{R} - {}^t_0\boldsymbol{F} \tag{13.9}$$

$$ {}^t_0\boldsymbol{K}_L = \int_{{}^0V} {}^t_0\boldsymbol{B}_L^T \, {}_0\boldsymbol{C} \, {}^t_0\boldsymbol{B}_L d^0V \tag{13.10}$$

$$ {}^t_0\boldsymbol{K}_{NL} = \int_{{}^0V} {}^t_0\boldsymbol{B}_{NL}^T \underbrace{{}^t_0\boldsymbol{S}}_{\text{matrix}} {}^t_0\boldsymbol{B}_{NL} d^0V \tag{13.11}$$

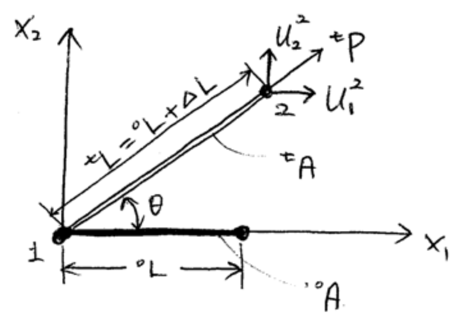
$$ {}^t_0\boldsymbol{F} = \int_{{}^0V} {}^t_0\boldsymbol{B}_L^T \underbrace{{}^t_0\hat{\boldsymbol{S}}}_{\text{vector}} d^0V \tag{13.12}$$

The iteration (full Newton-Raphson) is

$$ \left({}^{t+\Delta t}_{\ \ \ 0}\boldsymbol{K}_L^{(i-1)} + {}^{t+\Delta t}_{\ \ \ 0}\boldsymbol{K}_{NL}^{(i-1)} \right) \Delta\boldsymbol{U}^{(i)} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}_{\ \ \ 0}\boldsymbol{F}^{(i-1)} \tag{13.13}$$

$$ {}^{t+\Delta t}\boldsymbol{U}^{(i)} = {}^{t+\Delta t}\boldsymbol{U}^{(i-1)} + \Delta\boldsymbol{U}^{(i)} \tag{13.14}$$

Truss element example (p. 545)



Here we have to only deal with ${}^t_0S_{11}$, ${}_0e_{11}$, ${}_0\eta_{11}$

$$ {}_0e_{11} = \frac{\partial u_1}{\partial {}^0x_1} + \frac{\partial {}^tu_k}{\partial {}^0x_1} \cdot \frac{\partial u_k}{\partial {}^0x_1} \tag{13.15}$$

$$ {}_0\eta_{11} = \frac{1}{2}\left(\frac{\partial u_k}{\partial {}^0x_1} \cdot \frac{\partial u_k}{\partial {}^0x_1} \right) \tag{13.16}$$

We are after

$$ {}_0e_{11} = {}^t_0\boldsymbol{B}_L \begin{pmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \end{pmatrix} = {}^t_0\boldsymbol{B}_L \hat{\boldsymbol{u}} \tag{13.17}$$

$$ u_i = \sum_{k=1}^{2} h_k u_i^k \tag{13.18}$$

$$ {}^tu_i = \sum_{k=1}^{2} h_k \, {}^tu_i^k \tag{13.19}$$

$$
\begin{aligned}
{}_0e_{11} &= \frac{\partial u_1}{\partial {}^0x_1} + \frac{\partial {}^tu_1}{\partial {}^0x_1}\frac{\partial u_1}{\partial {}^0x_1} + \frac{\partial {}^tu_2}{\partial {}^0x_1}\frac{\partial u_2}{\partial {}^0x_1} && (13.20a)\\
{}^tu_1^2 &= \left({}^0L + \Delta L\right)\cos\theta - {}^0L && (13.20b)\\
{}^tu_2^2 &= \left({}^0L + \Delta L\right)\sin\theta && (13.20c)
\end{aligned}
$$

$$
\begin{aligned}
{}_0e_{11} =& \frac{1}{{}^0L}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\hat{\boldsymbol{u}}\\
&+ \left(\underbrace{\frac{{}^0L+\Delta L}{{}^0L}\cos\theta - 1}_{\frac{\partial {}^tu_1}{\partial {}^0x_1}}\right)\cdot\frac{1}{{}^0L}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\hat{\boldsymbol{u}}\\
&+ \left(\underbrace{\frac{{}^0L+\Delta L}{{}^0L}\sin\theta}_{\frac{\partial {}^tu_2}{\partial {}^0x_1}}\right)\cdot\frac{1}{{}^0L}\begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix}\hat{\boldsymbol{u}} && (13.20d)\\
=& {}_0^t\boldsymbol{B}_L\hat{\boldsymbol{u}} && (13.20e)
\end{aligned}
$$

Hence,

$$
{}_0e_{11} = \boxed{\frac{{}^0L+\Delta L}{\left({}^0L\right)^2}\begin{bmatrix} -\cos\theta & -\sin\theta & \cos\theta & \sin\theta \end{bmatrix}}\,\hat{\boldsymbol{u}} \qquad (13.20f)
$$

where the boxed quantity above equals ${}_0^t\boldsymbol{B}_L$. In small strain but large rotation analysis we assume $\Delta L \ll {}^0L$,

$$
{}_0e_{11} = \frac{1}{{}^0L}\begin{bmatrix} -\cos\theta & -\sin\theta & \cos\theta & \sin\theta \end{bmatrix}\hat{\boldsymbol{u}} \qquad (13.20g)
$$

$$
{}_0\eta_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial {}^0x_1}\frac{\partial u_1}{\partial {}^0x_1} + \frac{\partial u_2}{\partial {}^0x_1}\frac{\partial u_2}{\partial {}^0x_1}\right) \qquad (13.21a)
$$

$$
\begin{aligned}
\delta_0\eta_{11} &= \frac{1}{2}\left(\frac{\partial \delta u_1}{\partial {}^0x_1}\frac{\partial u_1}{\partial {}^0x_1} + \frac{\partial u_1}{\partial {}^0x_1}\frac{\partial \delta u_1}{\partial {}^0x_1} + \frac{\partial \delta u_2}{\partial {}^0x_1}\frac{\partial u_2}{\partial {}^0x_1} + \frac{\partial u_2}{\partial {}^0x_1}\frac{\partial \delta u_2}{\partial {}^0x_1}\right) && (13.21b)\\
&= \left(\frac{\partial \delta u_1}{\partial {}^0x_1}\frac{\partial u_1}{\partial {}^0x_1} + \frac{\partial \delta u_2}{\partial {}^0x_1}\frac{\partial u_2}{\partial {}^0x_1}\right) && (13.21c)
\end{aligned}
$$

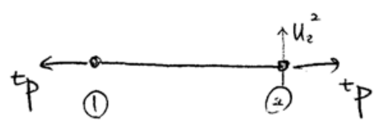
$$
{}_0^tS_{11}\delta_0\eta_{11} = \begin{bmatrix} \frac{\partial \delta u_1}{\partial {}^0x_1} & \frac{\partial \delta u_2}{\partial {}^0x_1} \end{bmatrix}\underbrace{\begin{pmatrix} {}_0^tS_{11} & 0 \\ 0 & {}_0^tS_{11} \end{pmatrix}}_{{}_0^t\boldsymbol{S}}\begin{pmatrix} \frac{\partial u_1}{\partial {}^0x_1} \\ \frac{\partial u_2}{\partial {}^0x_1} \end{pmatrix} \qquad (13.21d)
$$

$$
\begin{pmatrix} \frac{\partial u_1}{\partial {}^0x_1} \\ \frac{\partial u_2}{\partial {}^0x_1} \end{pmatrix} = \underbrace{\frac{1}{{}^0L}\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}}_{\boldsymbol{B}_{NL}}\hat{\boldsymbol{u}} \qquad (13.21e)
$$

$$ {}_0\boldsymbol{C} = E \tag{13.22} $$
$$ {}_0^t\hat{\boldsymbol{S}} = {}_0^tS_{11} \tag{13.23} $$

Assume small strains

$$ \begin{aligned} {}_0^t\boldsymbol{K} &= \frac{EA}{{}^0L} \\ &= \underbrace{\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ & & \cos^2\theta & \sin\theta\cos\theta \\ \text{sym} & & & \sin^2\theta \end{bmatrix}}_{{}_0^t\boldsymbol{K}_L} \\ &\quad + \underbrace{\frac{{}^tP}{{}^0L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}}_{{}_0^t\boldsymbol{K}_{NL}} \end{aligned} \tag{13.24} $$



When $\theta = 0$, ${}_0^t\boldsymbol{K}_L$ doesn't give stiffness corresponding to u_2^2, but ${}_0^t\boldsymbol{K}_{NL}$ does.

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Lecture 14 - Total Lagrangian formulation, cont'd

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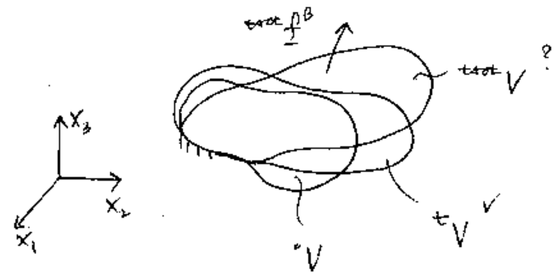
Truss element. 2D and 3D solids.



$$\int_{^{t+\Delta t}V} {}^{t+\Delta t}\tau_{ij}\delta_{t+\Delta t}e_{ij}d^{t+\Delta t}V = {}^{t+\Delta t}\mathcal{R} \tag{14.1}$$

$$\int_{^0V} {}^{t+\Delta t}_{\ \ \ 0}S_{ij}\delta^{t+\Delta t}_{\ \ \ 0}\epsilon_{ij}\delta^0V = {}^{t+\Delta t}\mathcal{R} \tag{14.2}$$

$$\Big\downarrow \text{ linearization}$$

$$\int_{^0V} {}_0C_{ijrs\,0}e_{rs}\delta_0e_{ij}\delta^0V + \int_{^0V} {}^t_0S_{ij}\delta_0\eta_{ij}\delta^0V = {}^{t+\Delta t}\mathcal{R} - \int_{^0V} {}^t_0S_{ij}\delta_0e_{ij}\delta^0V \tag{14.3}$$

Note:

$$\delta_0e_{ij} = \delta^t_0\epsilon_{ij}$$

varying with respect to the configuration at time t.

F.E. discretization

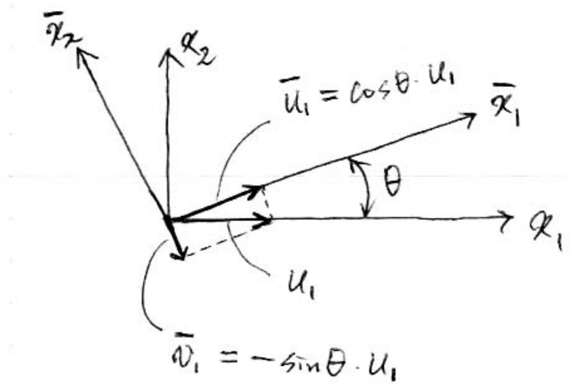
$$^0x_i = \sum_k h_k\,{}^0x_i^k \qquad {}^tx_i = \sum_k h_k\,{}^tx_i^k \qquad {}^{t+\Delta t}x_i = \sum_k h_k\,{}^{t+\Delta t}x_i^k \tag{14.4a}$$

$$^tu_i = \sum_k h_k\,{}^tu_i^k \qquad {}^{t+\Delta t}u_i = \sum_k h_k\,{}^{t+\Delta t}u_i^k \qquad u_i = \sum_k h_k u_i^k \tag{14.4b}$$

(14.4) into (14.3) gives

$$\left({}^t_0\boldsymbol{K}_L + {}^t_0\boldsymbol{K}_{NL}\right)\boldsymbol{U} = {}^{t+\Delta t}\boldsymbol{R} - {}^t_0\boldsymbol{F} \tag{14.5}$$

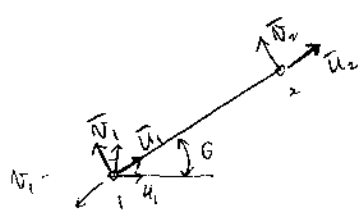
Truss



$\frac{\Delta L}{{}^0L} \ll 1$ small strain assumption:

$$
\begin{aligned}
{}_0^t\boldsymbol{K} =& \frac{E\,{}^0A}{{}^0L} \\
=& \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \sin\theta\cos\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \\
&+ \frac{{}^tP}{{}^0L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{14.6}
$$

(notice that the both matrices are symmetric)

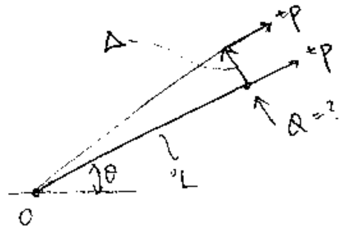


$$
\begin{pmatrix} \overline{u}_1 \\ \overline{v}_1 \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \tag{14.7}
$$

Corresponding to the $\overline{u}$ and $\overline{v}$ displacements we have:

$$
{}_0^t\boldsymbol{K} = \frac{E\,{}^0A}{{}^0L} \tag{14.8}
$$

$$
= \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{{}^tP}{{}^0L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{14.9}
$$



$$Q\,{}^0L = {}^tP \cdot \Delta \qquad \Rightarrow \qquad Q = \boxed{\frac{{}^tP}{{}^0L}} \cdot \Delta \tag{14.10}$$

where the boxed term is the stiffness. In axial direction, $\frac{{}^tP}{{}^0L}$ is not very important because usually $\frac{E\,{}^0A}{{}^0L} \gg \frac{{}^tP}{{}^0L}$. But, in vertical direction, $\frac{{}^tP}{{}^0L}$ is important.

$$
{}^t_0\boldsymbol{F} = {}^tP \begin{bmatrix} -\cos\theta \\ -\sin\theta \\ \cos\theta \\ \sin\theta \end{bmatrix} \tag{14.11}
$$

2D/3D (e.g. Table 6.5) *2D:*

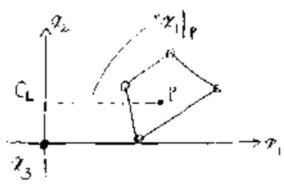
$$
{}_0\epsilon_{11} = \underbrace{{}_0u_{1,1} + {}^t_0u_{1,1}\,{}_0u_{1,1} + {}^t_0u_{2,1}\,{}_0u_{2,1}}_{{}_0e_{11}} + \frac{1}{2}\underbrace{\left[\left({}_0u_{1,1}\right)^2 + \left({}_0u_{2,1}\right)^2\right]}_{{}_0\eta_{11}} \tag{14.12}
$$

$$
{}_0\epsilon_{22} = \cdots \tag{14.13}
$$

$$
{}_0\epsilon_{12} = \cdots \tag{14.14}
$$

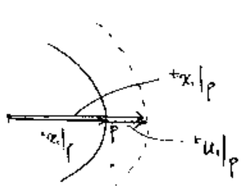
(Axisymmetric)

$$
{}_0\epsilon_{33} = \ ? \tag{14.15}
$$



$$
{}^t_0\boldsymbol{\epsilon} = \frac{1}{2}\left(\left({}^t_0\boldsymbol{U}\right)^2 - \boldsymbol{I}\right) \tag{14.16}
$$

$$
{}^t_0\boldsymbol{U}^2 = \begin{bmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \underset{\substack{\uparrow \\ ({}^t\lambda)^2}}{\times} \end{bmatrix} \tag{14.17}
$$



$$
\begin{aligned}
{}^t\lambda &= \frac{d\,{}^ts}{d\,{}^0s} = \frac{2\pi\left({}^0x_1 + {}^tu_1\right)}{2\pi\,{}^0x_1} \\
&= 1 + \frac{{}^tu_1}{{}^0x_1}
\end{aligned} \tag{14.18}
$$

$$
\begin{aligned}
{}^t_0\epsilon_{33} &= \frac{1}{2}\left[\left(1 + \frac{{}^tu_1}{{}^0x_1}\right)^2 - 1\right] \\
&= \frac{{}^tu_1}{{}^0x_1} + \frac{1}{2}\left(\frac{{}^tu_1}{{}^0x_1}\right)^2
\end{aligned} \tag{14.19}
$$

$$ {}^{t+\Delta t}_{\ \ \ 0}\epsilon_{33} = \frac{{}^{t}u_1 + u_1}{{}^{0}x_1} + \frac{1}{2}\left(\frac{{}^{t}u_1 + u_1}{{}^{0}x_1}\right)^2 \tag{14.20} $$

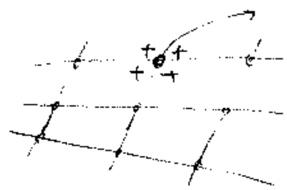
$$ {}_{0}\epsilon_{33} = {}^{t+\Delta t}_{\ \ \ 0}\epsilon_{33} - {}^{t}_{0}\epsilon_{33} = \frac{u_1}{{}^{0}x_1} + \frac{{}^{t}u_1}{{}^{0}x_1}\cdot\frac{u_1}{{}^{0}x_1} + \frac{1}{2}\left(\frac{u_1}{{}^{0}x_1}\right)^2 \tag{14.21} $$

How do we assess the accuracy of an analysis?

Reading: Sec. 4.3.6

- Mathematical model $\sim \boldsymbol{u}$
- F.E. solution $\sim \boldsymbol{u}_h$

Find $\|\boldsymbol{u} - \boldsymbol{u}_h\|$ and $\|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|$.



non-smoothed stress gives a good error measure.

References

[1] T. Sussman and K. J. Bathe. "Studies of Finite Element Procedures - on Mesh Selection." *Computers & Structures*, 21:257–264, 1985.

[2] T. Sussman and K. J. Bathe. "Studies of Finite Element Procedures - Stress Band Plots and the Evaluation of Finite Element Meshes." *Journal of Engineering Computations*, 3:178–191, 1986.

Lecture 15 - Field problems

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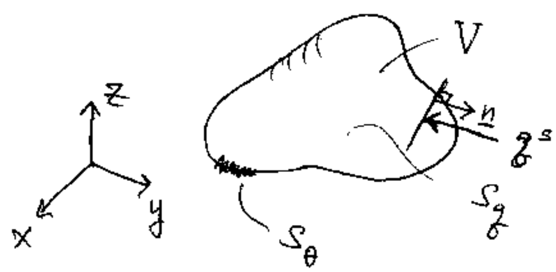
Heat transfer, incompressible/inviscid/irrotational flow, seepage flow, etc.

Reading: Sec. 7.2-7.3

- Differential formulation
- Variational formulation
- Incremental formulation
- F.E. discretization

15.1 Heat transfer



Assume V constant for now:

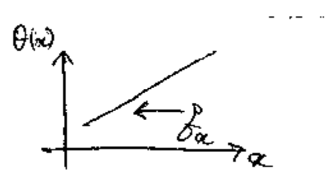
$$S = S_\theta \cup S_q$$

$\theta(x, y, z, t)$ is unkown except $\theta|_{S_\theta} = \theta_{pr}$. In addition, $q^s|_{S_q}$ is also prescribed.

15.1.1 Differential formulation

I. Heat flow equilibrium in V and on S_q.

II. Constitutive laws $q_x = -k\frac{\partial\theta}{\partial x}$.



$$q_y = -k\frac{\partial\theta}{\partial y} \tag{15.1}$$

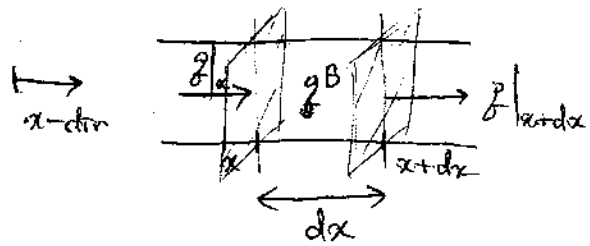
$$q_z = -k\frac{\partial\theta}{\partial z} \tag{15.2}$$

III. Compatibility: temperatures need to be continuous and satisfy the boundary conditions.

Heat flow equilibrium gives

$$\frac{\partial}{\partial x}\left(k\frac{\partial \theta}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial \theta}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial \theta}{\partial z}\right)=-q^B \tag{15.3}$$

where q^B is the heat generated per unit volume. Recall 1D case:



unit cross-section

$$dV = dx\cdot(1) \tag{15.4}$$

$$q|_x - q|_{x+dx} + q^B dx = 0 \tag{15.5}$$

$$q|_x - \left(q|_x + \frac{\partial q_x}{\partial x}dx\right) + q^B dx = 0 \tag{15.6}$$

$$-\frac{\partial}{\partial x}\left(-k\frac{\partial \theta}{\partial x}\right)\cancel{dx} + q^B \cancel{dx} = 0 \tag{15.7}$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial \theta}{\partial x}\right) = -q^B \tag{15.8}$$

We also need to satisfy

$$k\frac{\partial \theta}{\partial n} = q^S \tag{15.9}$$

on S_q.

15.1.2 Principle of virtual temperatures

$$\overline{\theta}\left(\frac{\partial}{\partial x}\left(k\frac{\partial \theta}{\partial x}\right)+\cdots+q^B\right)=0 \tag{15.10}$$

($\overline{\theta}\big|_{S_\theta}=0$ and $\overline{\theta}$ to be continuous.)

$$\int_V \overline{\theta}\left(\frac{\partial}{\partial x}\left(k\frac{\partial \theta}{\partial x}\right)+\cdots+q^B\right)dV = 0 \tag{15.11}$$

Transform using divergence theorem (see Ex 4.2, 7.1)

$$\int_V \overline{\boldsymbol{\theta}}'^T \underbrace{\boldsymbol{k}\boldsymbol{\theta}'}_{\text{heat flow}} dV = \int_V \overline{\theta} q^B dV + \int_{S_q} \overline{\theta}^{S_q} q^S dS_q \tag{15.12}$$

$$\boldsymbol{\theta}' = \begin{pmatrix} \frac{\partial \theta}{\partial x} \\ \\ \frac{\partial \theta}{\partial y} \\ \\ \frac{\partial \theta}{\partial z} \end{pmatrix} \tag{15.13}$$

$$\boldsymbol{k} = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} \tag{15.14}$$

Convection boundary condition

$$q^S = h\left(\theta^e - \theta^S\right) \tag{15.15}$$

where θ^e is the given environmental temperature.

Radiation

$$q^S = \kappa^* \left[(\theta^r)^4 - \left(\theta^S\right)^4\right] \tag{15.16}$$

$$= \kappa^* \left[(\theta^r)^2 + \left(\theta^S\right)^2\right] \left(\theta^r + \theta^S\right)\left(\theta^r - \theta^S\right) \tag{15.17}$$

$$= \kappa \left(\theta^r - \theta^S\right) \tag{15.18}$$

where $\kappa = \kappa(\theta^S)$ and θ^r is given temperature of source. At time $t + \Delta t$

$$\int_V \overline{\boldsymbol{\theta}}'^T \, {}^{t+\Delta t}\boldsymbol{k} \, {}^{t+\Delta t}\boldsymbol{\theta}' dV = \int_V \overline{\theta} \, {}^{t+\Delta t}q^B dV + \int_{S_q} \overline{\theta}^S \, {}^{t+\Delta t}q^S dS_q \tag{15.19}$$

$$\text{Let} \qquad {}^{t+\Delta t}\theta = {}^t\theta + \theta \tag{15.20}$$

$$\text{or} \qquad {}^{t+\Delta t}\theta^{(i)} = {}^{t+\Delta t}\theta^{(i-1)} + \Delta\theta^{(i)} \tag{15.21}$$

$$\text{with} \qquad {}^{t+\Delta t}\theta^{(0)} = {}^t\theta \tag{15.22}$$

From (15.19)

$$\begin{aligned} &\int_V \overline{\boldsymbol{\theta}}'^T \, {}^{t+\Delta t}\boldsymbol{k}^{(i-1)} \Delta\boldsymbol{\theta}'^{(i)} dV \\ &= \int_V \overline{\theta} \, {}^{t+\Delta t}q^B dV - \int_V \overline{\boldsymbol{\theta}}'^T \, {}^{t+\Delta t}\boldsymbol{k}^{(i-1)} \, {}^{t+\Delta t}\boldsymbol{\theta}'^{(i-1)} dV \\ &\quad + \int_{S_q} \boxed{\overline{\theta}^S \, {}^{t+\Delta t}h^{(i-1)}} \left({}^{t+\Delta t}\theta^e - \left({}^{t+\Delta t}\theta^{S^{(i-1)}} + \boxed{\Delta\theta^{S^{(i)}}} \right) \right) dS_q \end{aligned} \tag{15.23}$$

where the $\Delta\theta^{S^{(i)}}$ term would be moved to the left-hand side.

We considered the convection conditions

$$\int_{S_q} \overline{\theta}^S \, {}^{t+\Delta t}h \left({}^{t+\Delta t}\theta^e - {}^{t+\Delta t}\theta^S \right) dS_q \tag{15.24}$$

The radiation conditions would be included similarly.

F.E. discretization

$$ {}^{t+\Delta t}\theta = \boldsymbol{H}_{1\text{x}4} \cdot {}^{t+\Delta t}\hat{\boldsymbol{\theta}}_{4\text{x}1} \qquad \text{for 4-node 2D planar element} \tag{15.25}$$

$$ {}^{t+\Delta t}\boldsymbol{\theta}'_{2\text{x}1} = \boldsymbol{B}_{2\text{x}4} \cdot {}^{t+\Delta t}\hat{\boldsymbol{\theta}}_{4\text{x}1} \tag{15.26}$$

$$ {}^{t+\Delta t}\theta^S = \boldsymbol{H}^S \cdot {}^{t+\Delta t}\hat{\boldsymbol{\theta}} \tag{15.27}$$

For (15.23)

$$\int_V \overline{\boldsymbol{\theta}}'^T\, {}^{t+\Delta t}\boldsymbol{k}^{(i-1)} \Delta\boldsymbol{\theta}'^{(i)} dV \overset{\text{gives}}{\Longrightarrow} \left(\int_V \underbrace{\boldsymbol{B}^T}_{4\text{x}2} \underbrace{{}^{t+\Delta t}\boldsymbol{k}^{(i-1)}}_{2\text{x}2} \underbrace{\boldsymbol{B}}_{2\text{x}4} dV \right) \Delta \underbrace{\hat{\boldsymbol{\theta}}^{(i)}}_{4\text{x}1} \tag{15.28}$$

$$\int_V \overline{\theta}\, {}^{t+\Delta t}q^B dV \Rightarrow \int_V \boldsymbol{H}^T\, {}^{t+\Delta t}q^B dV \tag{15.29}$$

$$\int_V \overline{\boldsymbol{\theta}}'^T\, {}^{t+\Delta t}\boldsymbol{k}^{(i-1)}\, {}^{t+\Delta t}\boldsymbol{\theta}'^{(i-1)} dV \Rightarrow \left(\int_V \boldsymbol{B}^T\, {}^{t+\Delta t}\boldsymbol{k}^{(i-1)} \boldsymbol{B} dV \right) \underbrace{{}^{t+\Delta t}\hat{\boldsymbol{\theta}}^{(i-1)}}_{\text{known}} \tag{15.30}$$

$$\begin{aligned} &\int_{S_q} \overline{\theta}^{S^T}\, {}^{t+\Delta t}h^{(i-1)} \left({}^{t+\Delta t}\theta^e - \left({}^{t+\Delta t}\theta^{S^{(i-1)}} + \Delta\theta^{S^{(i)}} \right) \right) dS_q \quad \Longrightarrow \\ &\quad \int_{S_q} \underbrace{\boldsymbol{H}^{S^T}}_{4\text{x}1}\, {}^{t+\Delta t}h^{(i-1)} \underbrace{\boldsymbol{H}^S}_{1\text{x}4} \left(\underbrace{{}^{t+\Delta t}\hat{\boldsymbol{\theta}}^e}_{4\text{x}1} - \left(\underbrace{{}^{t+\Delta t}\hat{\boldsymbol{\theta}}^{(i-1)}}_{4\text{x}1} + \Delta \underbrace{\hat{\boldsymbol{\theta}}^{(i)}}_{4\text{x}1} \right) \right) dS_q \end{aligned} \tag{15.31}$$

15.2 Inviscid, incompressible, irrotational flow

2D case: v_x, v_y are velocities in x and y directions.

Reading: Sec. 7.3.2

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \tag{15.32}$$

or

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \qquad \text{(incompressible)} \tag{15.33}$$

$$\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} = 0 \qquad \text{(irrotational)} \tag{15.34}$$

Use the potential $\phi(x, y)$,

$$v_x = \frac{\partial \phi}{\partial x} \qquad v_y = \frac{\partial \phi}{\partial y} \tag{15.35}$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{in } V \tag{15.36}$$

(Same as the heat transfer equation with $k = 1$, $q^B = 0$)

2.094 — Finite Element Analysis of Solids and Fluids Fall '08

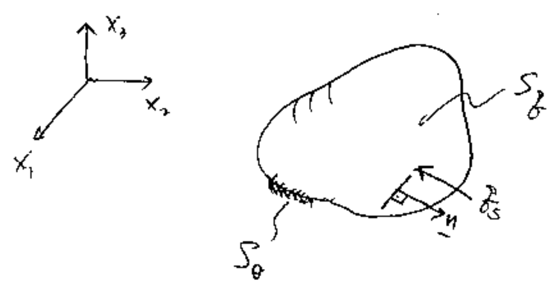
Lecture 16 - F.E. analysis of Navier-Stokes fluids

Prof. K.J. Bathe MIT OpenCourseWare

Incompressible flow with heat transfer

Reading: Sec. 7.1-7.4, Table 7.3

We recall heat transfer for a solid:



Governing differential equations

$$(k\theta_{,i})_{,i} + q^B = 0 \qquad \text{in } V \tag{16.1}$$

$$\theta\Big|_{S_\theta} \text{ is prescribed, } k\frac{\partial\theta}{\partial n}\Big|_{S_q} = q^S\Big|_{S_q} \tag{16.2}$$

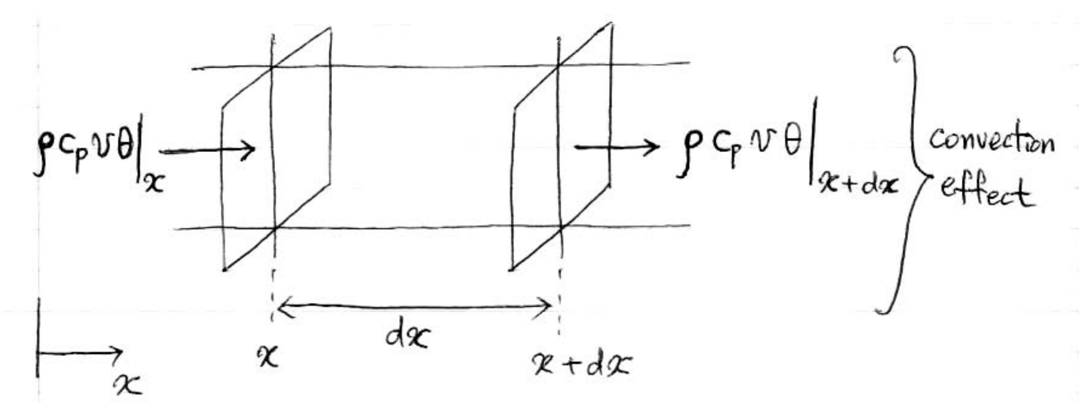
$$\mathrm{S}_\theta \cup \mathrm{S}_q = \mathrm{S} \qquad \mathrm{S}_\theta \cap \mathrm{S}_q = \varnothing \tag{16.3}$$

Principle of virtual temperatures

$$\int_V \overline{\theta}_{,i} k\theta_{,i} dV = \int_V \overline{\theta} q^B dV + \int_{S_q} \overline{\theta}^S q^S dS_q \tag{16.4}$$

for arbitrary continuous $\overline{\theta}(x_1, x_2, x_3)$ zero on S_θ

For a fluid, we use the Eulerian formulation.



$$\rho c_p v\ \theta|_x - \left\{ \rho c_p v\ \theta|_x + \frac{\partial}{\partial x}(\rho c_p v \theta) dx \right\} + \text{ conduction } + \text{ etc} \tag{16.5}$$

In general 3D, we have an additional term for the left hand side of (16.1):

$$-\boldsymbol{\nabla} \cdot (\rho c_p \boldsymbol{v} \theta) = -\rho c_p \boldsymbol{\nabla} \cdot (\boldsymbol{v}\theta) = -\rho c_p (\cancel{\boldsymbol{\nabla} \cdot \boldsymbol{v}})\theta - \underbrace{\rho c_p \left(\boldsymbol{v} \cdot \boldsymbol{\nabla} \right) \theta}_{\text{term (A)}} \tag{16.6}$$

where $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ in the incompressible case.

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = v_{i,i} = \text{div}(\boldsymbol{v}) = 0 \tag{16.7}$$

So (16.1) becomes

$$\left(k\theta_{,i}\right)_{,i} + q^B = \rho c_p \theta_{,i} v_i \Rightarrow \left(k\theta_{,i}\right)_{,i} + \left(q^B - \rho c_p \theta_{,i} v_i\right) = 0 \tag{16.8}$$

Principle of virtual temperatures is now (use (16.4))

$$\int_V \overline{\theta}_{,i} k \theta_{,i} dV + \int_V \overline{\theta} \left(\rho c_p \theta_{,i} v_i \right) dV = \int_V \overline{\theta} q^B dV + \int_{S_q} \overline{\theta}^S q^S dS_q \tag{16.9}$$

Navier-Stokes equations

- *Differential form*

$$\tau_{ij,j} + f_i^B = \rho v_{i,j} v_j \tag{16.10}$$

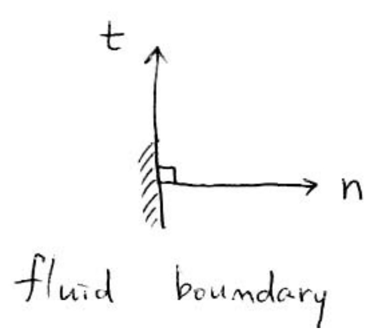
with $\rho v_{i,j} v_j$ like term (A) in (16.6) $= \rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$ in V.

$$\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij} \qquad e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \tag{16.11}$$

- *Boundary conditions* (need be modified for various flow conditions)

$$\tau_{ij} n_j = f_i^{S_f} \text{ on } S_f \tag{16.12}$$

Mostly used as $f_n = \tau_{nn} =$ prescribed, $f_t =$ unknown with possibly $\frac{\partial v_n}{\partial n} = \frac{\partial v_t}{\partial n} = 0$ (outflow or inflow conditions).



And v_i prescribed on S_v, and $\mathrm{S}_v \cup \mathrm{S}_f = \mathrm{S}$ and $\mathrm{S}_v \cap \mathrm{S}_f = \varnothing$.

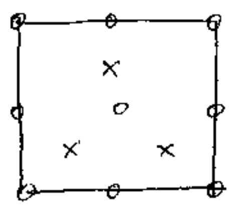
- *Variational form*

$$\int_V \overline{v}_i \rho v_{i,j} v_j dV + \int_V \overline{e}_{ij} \tau_{ij} dV = \int_V \overline{v}_i f_i^B dV + \int_{S_f} \overline{v}_i^{S_f} f_i^{S_f} dS_f \tag{16.13}$$

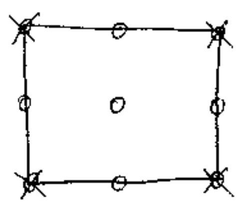
$$\int_V \overline{p} \boldsymbol{\nabla} \cdot \boldsymbol{v} dV = 0 \tag{16.14}$$

- *F.E. solution*

  We interpolate (x_1, x_2, x_3), v_i, $\overline{v}_i$, θ, $\overline{\theta}$, p, $\overline{p}$. Good elements are



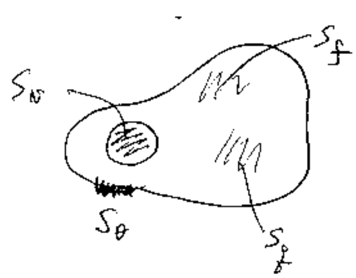
$\times$: linear pressure
$\circ$: biquadratic velocities
(Q_2, P_1), 9/3 element



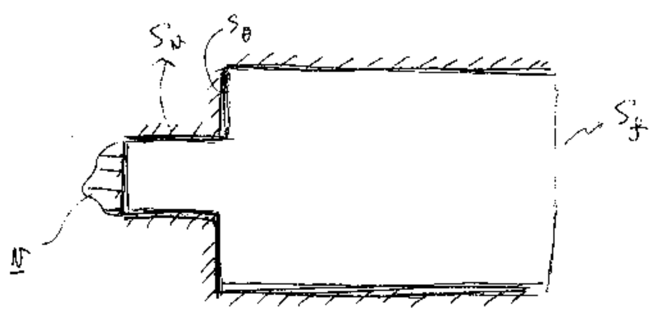
9/4c element

  Both satisfy the inf-sup condition.

So in general,



Example:

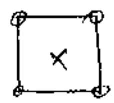


For S_f e.g.

$$\tau_{nn} = 0, \qquad \frac{\partial v_t}{\partial n} = 0; \tag{16.15}$$

and $\frac{\partial v_n}{\partial t}$ is solved for. Actually, we frequently just set $p = 0$.

Frequently used is the 4-node element with constant pressure



It does not strictly satisfy the inf-sup condition. Or use

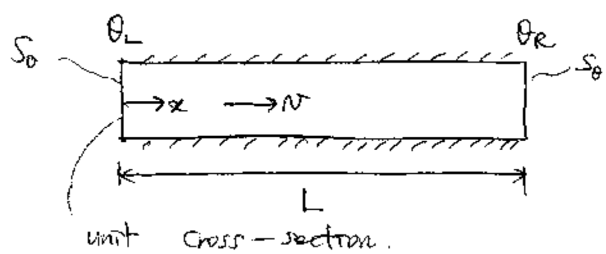
Reading: Sec. 7.4

3-node element with a bubble node.
Satisfies inf-sup condition

1D case of heat transfer with fluid flow, $v =$ constant

Reading: Sec. 7.4.3

$$\mathrm{Re} = \frac{vL}{\nu} \qquad \mathrm{Pe} = \frac{vL}{\alpha} \qquad \alpha = \frac{k}{\rho c_p} \tag{16.16}$$



$S_q = \phi$.

- *Differential equations*

$$k\theta'' = \rho c_p \theta' v \tag{16.17}$$

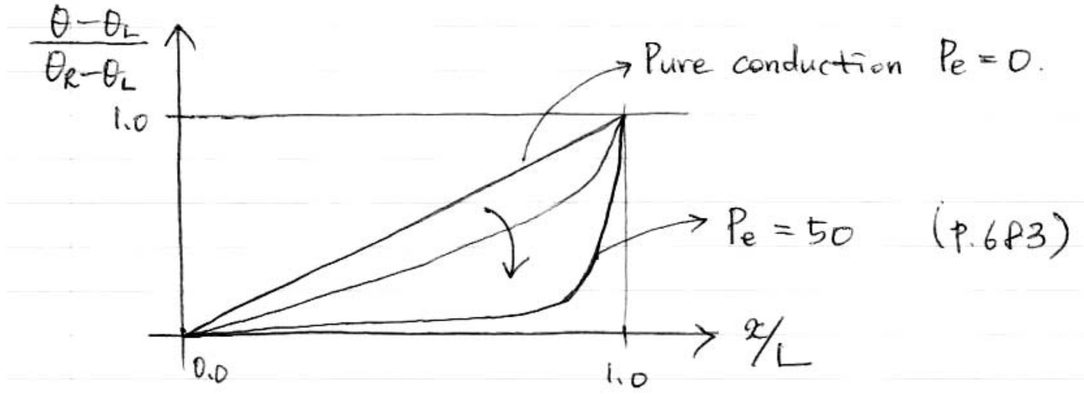
$$\theta|_{x=0} = \theta_L \qquad \theta|_{x=L} = \theta_R \tag{16.18}$$

In non-dimensional form

Reading: p. 683

$$\boxed{\frac{1}{\mathrm{Pe}}\theta'' = \theta'} \qquad \text{(now } \theta'' \text{ and } \theta' \text{ are non-dimensional)} \tag{16.19}$$

$$\Rightarrow \frac{\theta - \theta_L}{\theta_R - \theta_L} = \frac{\exp\left(\frac{\mathrm{Pe}}{L}x\right) - 1}{\exp(\mathrm{Pe}) - 1} \tag{16.20}$$

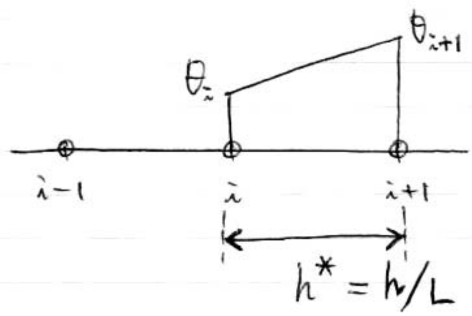


- *F.E. discretization*

$$\theta'' = \mathrm{Pe}\theta' \tag{16.21}$$

$$\int_0^1 \overline{\theta}' \theta' dx + \mathrm{Pe} \int_0^1 \overline{\theta}\theta' dx = 0 + \{ \text{ effect of boundary conditions } = 0 \text{ here}\} \tag{16.22}$$

Using 2-node elements gives



$$\frac{1}{(h^*)^2}\left(\theta_{i+1} - 2\theta_i + \theta_{i-1}\right) = \frac{\mathrm{Pe}}{2h^*}\left(\theta_{i+1} - \theta_{i-1}\right) \tag{16.23}$$

$$\mathrm{Pe} = \frac{vL}{\alpha} \tag{16.24}$$

Define

$$\mathrm{Pe}^e = \mathrm{Pe} \cdot \frac{h}{L} = \frac{vh}{\alpha} \tag{16.25}$$

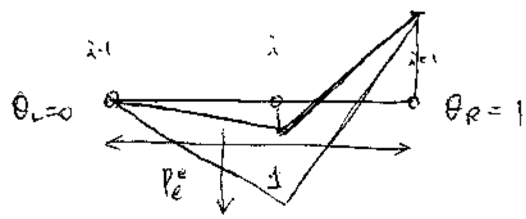
$$\left(-1 - \frac{\mathrm{Pe}^e}{2}\right)\theta_{i-1} + 2\theta_i + \left(\frac{\mathrm{Pe}^e}{2} - 1\right)\theta_{i+1} = 0 \tag{16.26}$$

what is happening when Pe^e is large? Assume two 2-node elements only.

$$\theta_{i-1} = 0 \tag{16.27}$$

$$\theta_{i+1} = 1 \tag{16.28}$$

$$\theta_i = \frac{1}{2}\left(1 - \frac{\mathrm{Pe}^e}{2}\right) \tag{16.29}$$



$$\theta_i = \frac{1}{2}\left(1 - \frac{\mathrm{Pe}^e}{2}\right) \tag{16.30}$$

For $\mathrm{Pe}^e > 2$, we have *negative* θ_i (unreasonable).

2.094 — Finite Element Analysis of Solids and Fluids	Fall '08
Lecture 17 - Incompressible fluid flow and heat transfer, cont'd	
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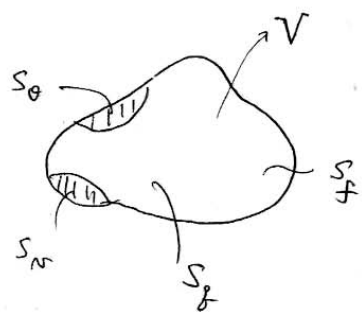
17.1 Abstract body

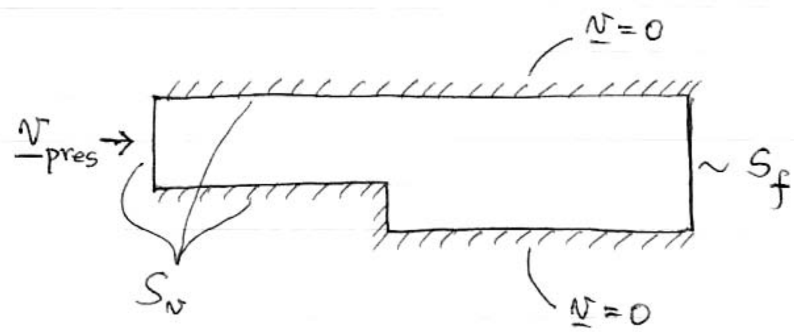
Reading:
Sec. 7.4

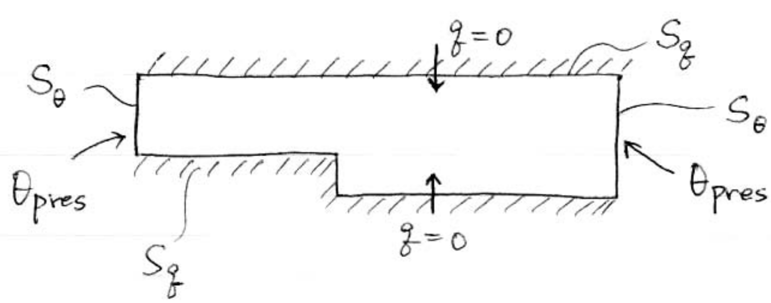


Fluid Flow
S_v, S_f
$S_v \cup S_f = S$
$S_v \cap S_f = 0$

Heat transfer
S_θ, S_q
$S_\theta \cup S_q = S$
$S_\theta \cap S_q = 0$

17.2 Actual 2D problem (channel flow)





17.3 Basic equations

P.V. velocities

$$\int_V \overline{v}_i \rho v_{i,j} v_j dV + \int_V \tau_{ij} \overline{e}_{ij} dV = \int_V \overline{v}_i f_i^B dV + \int_{S_f} \overline{v}_i^{S_f} f_i^{S_f} dS_f \tag{17.1}$$

Continuity

$$\int_V \overline{p} v_{i,i} dV = 0 \tag{17.2}$$

P.V. temperature

$$\int_V \overline{\theta} \rho c_p \theta_{,i} v_i dV + \int_V \overline{\theta}_{,i} k \theta_{,i} dV = \int_V \overline{\theta} q^B dV + \int_{S_q} \overline{\theta}^S q^S dS \tag{17.3}$$

F.E. solution

$$x_i = \sum h_k x_i^k \tag{17.4}$$

$$v_i = \sum h_k v_i^k \tag{17.5}$$

$$\theta = \sum h_k \theta_k \tag{17.6}$$

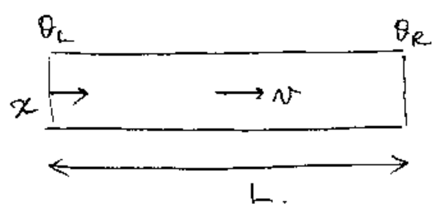
$$p = \sum \tilde{h}_k p_k \tag{17.7}$$

$$\Rightarrow \boxed{\boldsymbol{F}(\boldsymbol{u}) = \boldsymbol{R}} \qquad \boldsymbol{u} = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{p} \\ \boldsymbol{\theta} \end{pmatrix} \text{ nodal variables} \tag{17.8}$$

17.4 Model problem

1D equation,

$$\rho c_p v \frac{d\theta}{dx} = k \frac{d^2\theta}{dx^2} \tag{17.9}$$

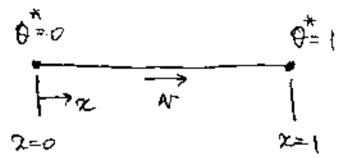


(v is given, unit cross section)

Non-dimensional form (Section 7.4)

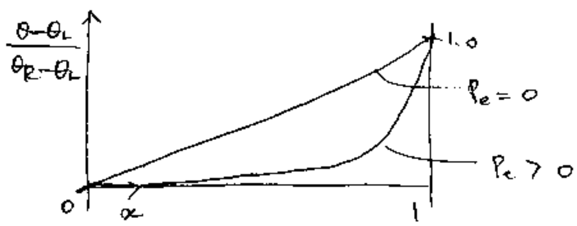
$$\boxed{\mathrm{Pe} \frac{d\theta}{dx} = \frac{d^2\theta}{dx^2}} \tag{17.10}$$

$$\mathrm{Pe} = \frac{vL}{\alpha}, \qquad \alpha = \frac{k}{\rho c_p} \tag{17.11}$$



θ^* is non-dimensional

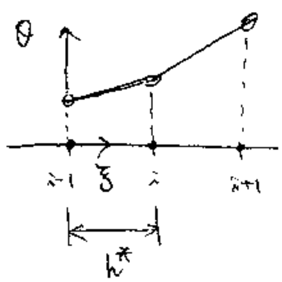
$$\frac{\theta - \theta_L}{\theta_R - \theta_L} = \frac{\exp\left(\frac{\mathrm{Pe}\, x}{L}\right) - 1}{\exp\left(\mathrm{Pe}\right) - 1} \tag{17.12}$$



(17.10) in F.E. analysis becomes

$$\int_V \overline{\theta} \mathrm{Pe} \frac{d\theta}{dx} dV + \int_V \frac{d\overline{\theta}}{dx} \frac{d\theta}{dx} dV = 0 \tag{17.13}$$

Discretized by linear elements:



$$h^* = \frac{h}{L}$$

$$\theta(\xi) = \left(1 - \frac{\xi}{h^*}\right) \theta_{i-1} + \frac{\xi}{h^*} \theta_i \tag{17.14}$$

For node i:

$$-\theta_{i-1} - \frac{\mathrm{Pe}^e}{2} \theta_{i-1} + 2\theta_i - \theta_{i+1} + \frac{\mathrm{Pe}^e}{2} \theta_{i+1} = 0 \tag{17.15}$$

where

$$\mathrm{Pe}^e = \frac{vh}{\alpha} \qquad \left(= \mathrm{Pe} \frac{h}{L}\right) \tag{17.16}$$

This result is the same as obtained by finite differences

$$\theta'' \Big|_i = \frac{1}{(h^*)^2} \left(\theta_{i+1} - 2\theta_i + \theta_{i-1}\right) \tag{17.17}$$

$$\theta' \Big|_i = \frac{\theta_{i+1} - \theta_{i-1}}{2h^*} \tag{17.18}$$

Considered $\theta_{i+1} = 1$, $\theta_{i-1} = 0$. Then

$$\theta_i = \frac{1 - (\mathrm{Pe}^e/2)}{2} \tag{17.19}$$

Physically unrealistic solution when $\mathrm{Pe}^e > 2$. For this not to happen, we should refine the mesh—a very fine mesh would be required. We use "upwinding"

$$\left.\frac{d\theta}{dx}\right|_i = \frac{\theta_i - \theta_{i-1}}{h^*} \tag{17.20}$$

The result is

$$\left(-1 - \mathrm{Pe}^e\right)\theta_{i-1} + \left(2 + \mathrm{Pe}^e\right)\theta_i - \theta_{i+1} = 0 \tag{17.21}$$

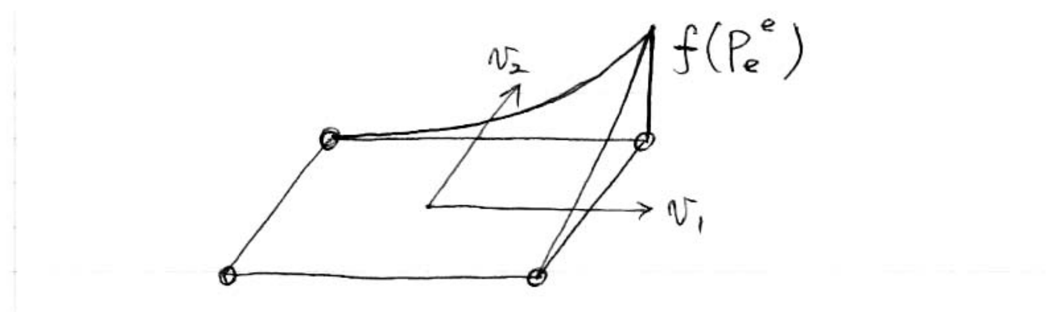
Very stable, e.g.

$$\left.\begin{aligned}\theta_{i-1} &= 0\\ \theta_{i+1} &= 1\end{aligned}\right\} \Rightarrow \theta_i = \frac{1}{2 + \mathrm{Pe}^e} \tag{17.22}$$

Unfortunately it is not that accurate. To obtain better accuracy in the interpolation for θ, use the function

$$\frac{\exp\left(\mathrm{Pe}\frac{x}{L}\right) - 1}{\exp\left(\mathrm{Pe}\right) - 1} \tag{17.23}$$

The result is Pe^e dependent:



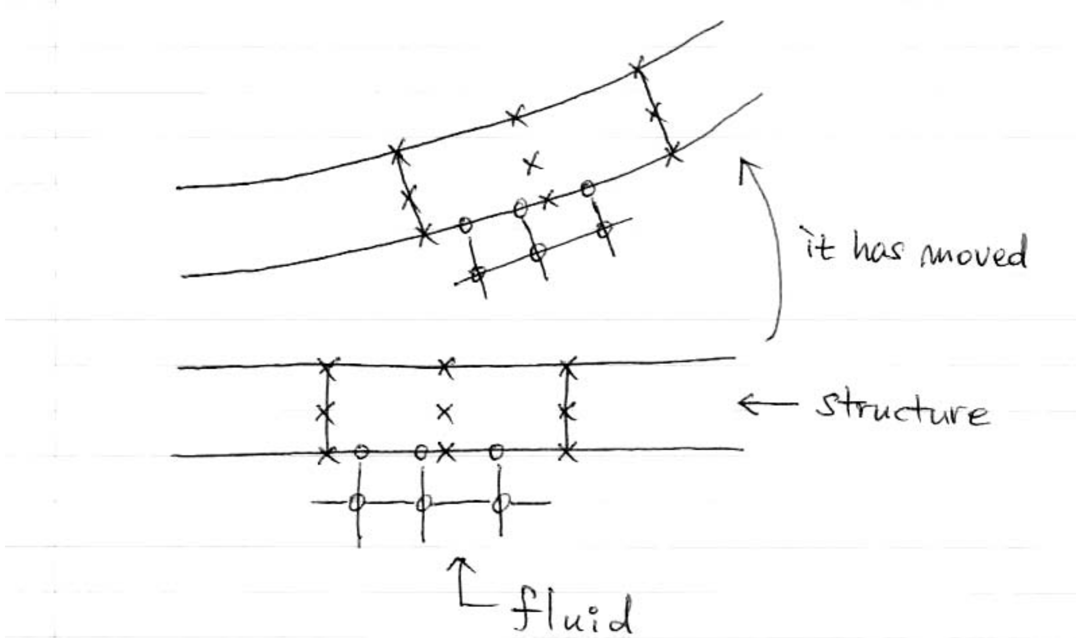
This implies flow-condition based interpolation. We use such interpolation functions—see references.

References

[1] K.J. Bathe and H. Zhang. "A Flow-Condition-Based Interpolation Finite Element Procedure for Incompressible Fluid Flows." *Computers & Structures*, 80:1267–1277, 2002.

[2] H. Kohno and K.J. Bathe. "A Flow-Condition-Based Interpolation Finite Element Procedure for Triangular Grids." *International Journal for Numerical Methods in Fluids*, 51:673–699, 2006.

17.5 FSI briefly



Lagrangian formulation for the structure/solid

Arbitrary Lagrangian-Eulerian (ALE) formulation Let f be a variable of a particle (e.g. $f = \theta$). Consider 1D

$$\dot{f}\Big|_{\text{particle}} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} v \tag{17.24}$$

where v is the particle velocity. For a mesh point,

$$f^*\Big|_{\text{mesh point}} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} v_m \tag{17.25}$$

where v_m is the mesh point velocity. Hence,

$$\dot{f}\Big|_{\text{particle}} = f^*\Big|_{\text{mesh point}} + \frac{\partial f}{\partial x}(v - v_m) \tag{17.26}$$

Use (17.26) in the momentum and energy equations and use force equilibrium and compatibility at the FSI boundary to set up the governing F.E. equations.

References

[1] K.J. Bathe, H. Zhang and M.H. Wang. "Finite Element Analysis of Incompressible and Compressible Fluid Flows with Free Surfaces and Structural Interactions." *Computers & Structures*, 56:193–213, 1995.

[2] K.J. Bathe, H. Zhang and S. Ji. "Finite Element Analysis of Fluid Flows Fully Coupled with Structural Interactions." *Computers & Structures*, 72:1–16, 1999.

[3] K.J. Bathe and H. Zhang. "Finite Element Developments for General Fluid Flows with Structural Interactions." *International Journal for Numerical Methods in Engineering*, 60:213–232, 2004.

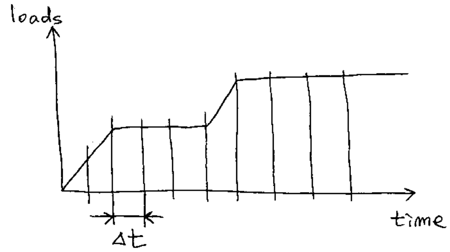
Lecture 18 - Solution of F.E. equations

Prof. K.J. Bathe MIT OpenCourseWare

In structures,

Reading: Sec. 8.4

$$\boldsymbol{F}(\boldsymbol{u}, \boldsymbol{p}) = \boldsymbol{R}. \tag{18.1}$$



In heat transfer,

$$\boldsymbol{F}(\boldsymbol{\theta}) = \boldsymbol{Q} \tag{18.2}$$

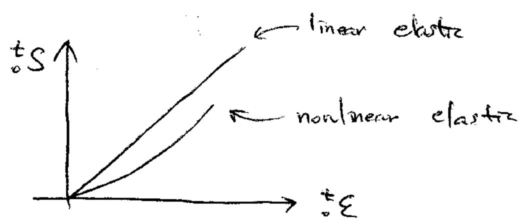
In fluid flow,

$$\boldsymbol{F}(\boldsymbol{v}, \boldsymbol{p}, \boldsymbol{\theta}) = \boldsymbol{R} \tag{18.3}$$

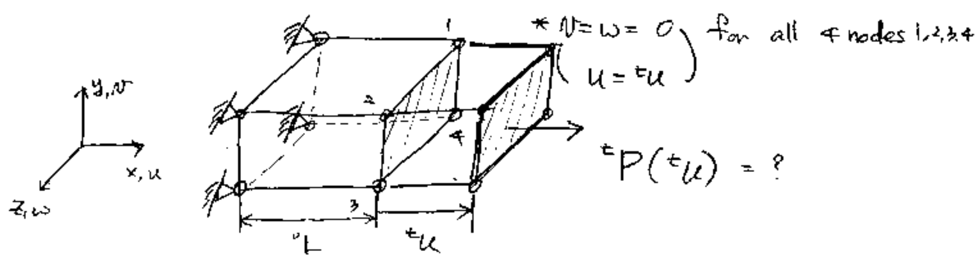
In structures/solids

$$\boldsymbol{F} = \sum_m \boldsymbol{F}^{(m)} = \sum_m \int_{{}^0V^{(m)}} {}_0^t\boldsymbol{B}_L^{(m)^T} \, {}_0^t\hat{\boldsymbol{S}}^{(m)} d^0V^{(m)} \tag{18.4}$$

Elastic materials



Example p. 590 textbook



Material law

$$ {}_0^tS_{11} = \tilde{E}\, {}_0^t\epsilon_{11} \tag{18.5} $$

In isotropic elasticity:

$$ \tilde{E} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, \qquad (\nu = 0.3) \tag{18.6} $$

$$ {}_0^t\boldsymbol{\epsilon} = \frac{1}{2}\left[\left({}_0^t\boldsymbol{U}\right)^2 - \boldsymbol{I}\right] \Rightarrow {}_0^t\epsilon_{11} = \frac{1}{2}\left[\left(\frac{{}^0L + {}^tu}{{}^0L}\right)^2 - 1\right] = \frac{1}{2}\left[\left(1 + \frac{{}^tu}{{}^0L}\right)^2 - 1\right] \tag{18.7} $$

where ${}_0^t\boldsymbol{U}$ is the stretch tensor.

$$ {}_0^tS_{11} = \frac{{}^0\rho}{{}^t\rho}\, {}_t^0X_{11}\, {}^t\tau_{11}\, {}_t^0X_{11}^T \tag{18.8} $$

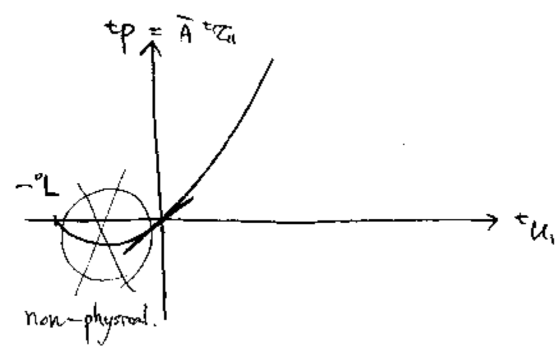
with

$$ {}_t^0X_{11} = \frac{{}^0L}{{}^0L + {}^tu}, \qquad {}^0\rho\, {}^0L = {}^t\rho\, {}^tL \tag{18.9} $$

$$ \Rightarrow {}_0^tS_{11} = \frac{{}^tL}{{}^0L}\left(\frac{{}^0L}{{}^tL}\right)^2 {}^t\tau_{11} = \frac{{}^0L}{{}^tL}\, {}^t\tau_{11} \tag{18.10} $$

$$ \therefore \frac{{}^0L}{{}^tL}\, {}^t\tau_{11} = \tilde{E} \cdot \frac{1}{2}\left[\left(1 + \frac{{}^tu}{{}^0L}\right)^2 - 1\right] \tag{18.11} $$

$$ \Rightarrow {}^t\tau_{11}\overline{A} = \boxed{{}^tP = \frac{\tilde{E}\overline{A}}{2}\left[\left(1 + \frac{{}^tu}{{}^0L}\right)^2 - 1\right]\left(1 + \frac{{}^tu}{{}^0L}\right)} \tag{18.12} $$



This is because of the material-law assumption (18.5) (okay for small strains ...)

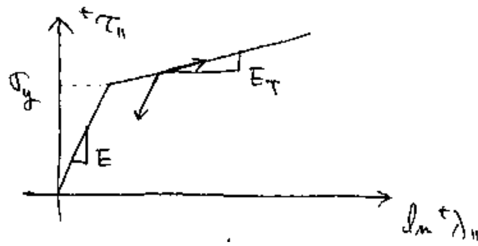
Hyperelasticity

$$ {}_0^tW = f(\text{Green-Lagrange strains, material constants}) \tag{18.13} $$

$$ {}_0^tS_{ij} = \frac{1}{2}\left(\frac{\partial\, {}_0^tW}{\partial\, {}_0^t\epsilon_{ij}} + \frac{\partial\, {}_0^tW}{\partial\, {}_0^t\epsilon_{ji}}\right) \tag{18.14} $$

$$ {}_0C_{ijrs} = \frac{1}{2}\left(\frac{\partial\, {}_0^tS_{ij}}{\partial\, {}_0^t\epsilon_{rs}} + \frac{\partial\, {}_0^tS_{ij}}{\partial\, {}_0^t\epsilon_{sr}}\right) \tag{18.15} $$

Plasticity



- yield criterion
- flow rule
- hardening rule

$$ {}^{t}\boldsymbol{\tau} = {}^{t-\Delta t}\boldsymbol{\tau} + \int_{t-\Delta t}^{t} d\boldsymbol{\tau} \tag{18.16}$$

Solution of (18.1) (similarly (18.2) and (18.3))

Newton-Raphson Find $\boldsymbol{U}^*$ as the zero of $f(\boldsymbol{U}^*)$

$$\boldsymbol{f}(\boldsymbol{U}^*) = {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F} \tag{18.17}$$

$$= \boldsymbol{f}\left({}^{t+\Delta t}\boldsymbol{U}^{(i-1)}\right) + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{U}}\right|_{{}^{t+\Delta t}\boldsymbol{U}^{(i-1)}} \cdot \left(\boldsymbol{U}^* - {}^{t+\Delta t}\boldsymbol{U}^{(i-1)}\right) + H.O.T. \tag{18.18}$$

where ${}^{t+\Delta t}\boldsymbol{U}^{(i-1)}$ is the value we just calculated and an approximation to $\boldsymbol{U}^*$.

Assume ${}^{t+\Delta t}\boldsymbol{R}$ is independent of the displacements.

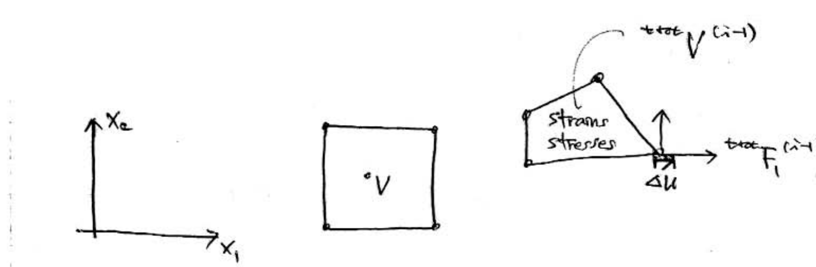
$$\boldsymbol{0} = \left({}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F}^{(i-1)}\right) - \left.\frac{\partial\, {}^{t+\Delta t}\boldsymbol{F}}{\partial \boldsymbol{U}}\right|_{{}^{t+\Delta t}\boldsymbol{U}^{(i-1)}} \cdot \Delta \boldsymbol{U}^{(i)} \tag{18.19}$$

We obtain

$$ {}^{t+\Delta t}\boldsymbol{K}^{(i-1)} \Delta \boldsymbol{U}^{(i)} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F}^{(i-1)} \tag{18.20}$$

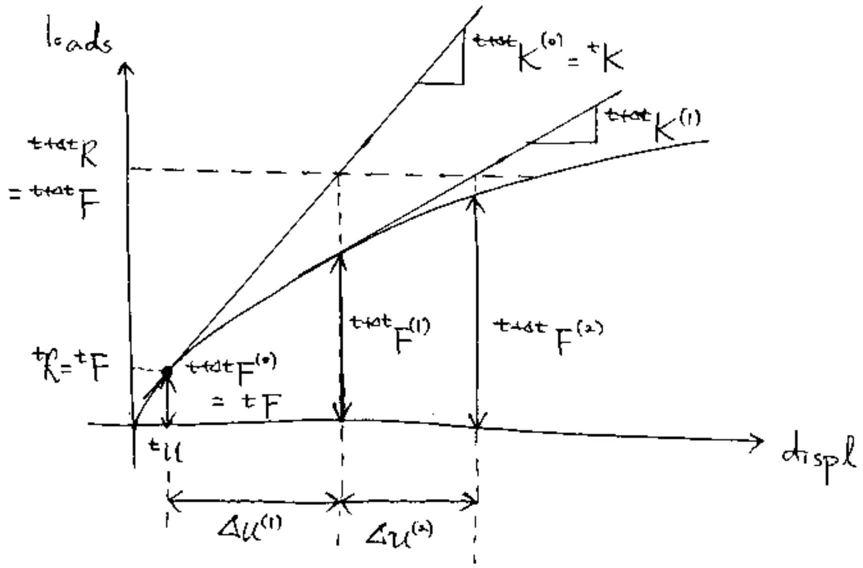
$$ {}^{t+\Delta t}\boldsymbol{K}^{(i-1)} = \left.\frac{\partial\, {}^{t+\Delta t}\boldsymbol{F}}{\partial \boldsymbol{U}}\right|_{{}^{t+\Delta t}\boldsymbol{U}^{(i-1)}} = \left.\left(\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}\right)\right|_{{}^{t+\Delta t}\boldsymbol{U}^{(i-1)}} \tag{18.21}$$

Physically



$$ {}^{t+\Delta t}K_{11}^{(i-1)} = \frac{\Delta\left({}^{t+\Delta t}F_1^{(i-1)}\right)}{\Delta u} \tag{18.22}$$

Pictorially for a single degree of freedom system



$$i = 1; \qquad {}^tK\Delta u^{(1)} = {}^{t+\Delta t}R - {}^tF \tag{18.23}$$

$$i = 2; \qquad {}^{t+\Delta t}K^{(1)}\Delta u^{(2)} = {}^{t+\Delta t}R - {}^{t+\Delta t}F^{(1)} \tag{18.24}$$

Convergence Use

$$\|\Delta \boldsymbol{U}^{(i)}\|_2 < \epsilon \tag{18.25}$$

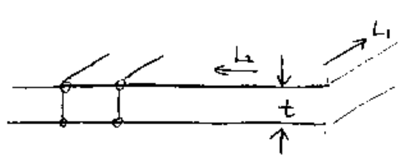
$$\|\boldsymbol{a}\|_2 = \sqrt{\sum_i (a_i)^2} \tag{18.26}$$

But, if incremental displacements are small in every iteration, need to also use

$$\|{}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F}^{(i-1)}\|_2 < \epsilon_R \tag{18.27}$$

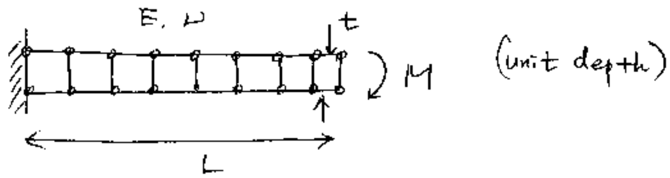
18.1 Slender structures

(beams, plates, shells)

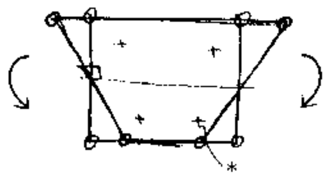


$$\frac{t}{L_i} \ll 1 \tag{18.28}$$

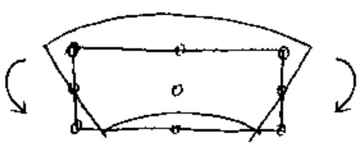
Beam



e.g. $\frac{t}{L} = \frac{1}{100}$



(4-node el.)
The element does not have curvature → we have a spurious shear strain



(9-node el.)
→ We do not have a shear (better)
→ But, still for thin structures, it has problems like ill-conditioning.

⇒ We need to use beam elements. For curved structures also spurious membrane strain can be present.

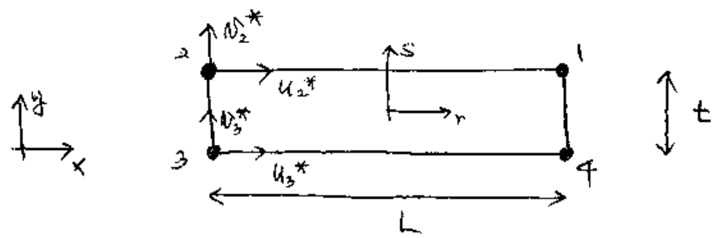
2.094 — Finite Element Analysis of Solids and Fluids Fall '08

Lecture 19 - Slender structures

Prof. K.J. Bathe MIT OpenCourseWare

Beam analysis, $\frac{t}{L} \ll 1$ (e.g. $\frac{t}{L} = \frac{1}{100}, \frac{1}{1000}, \cdots$)

Reading: Sec. 5.4, 6.5

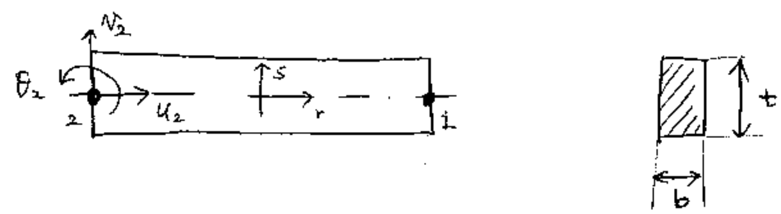


(plane stress)

$$\boldsymbol{J} = \begin{bmatrix} \frac{L}{2} & 0 \\ 0 & \frac{t}{2} \end{bmatrix} \tag{19.1}$$

$$h_2 = \frac{1}{4}(1-r)(1+s) \tag{19.2}$$

$$h_3 = \frac{1}{4}(1-r)(1-s) \tag{19.3}$$



Beam theory assumptions (Timoshenko beam theory):

$$v_2^* = v_3^* = v_2 \tag{19.4}$$

$$u_3^* = u_2 + \frac{t}{2}\theta_2 \tag{19.5}$$

$$u_2^* = u_2 - \frac{t}{2}\theta_2 \tag{19.6}$$

$$\begin{array}{cc} & \begin{array}{cccc} \quad u_2^* \quad & \quad v_2^* \quad & \quad u_3^* \quad & \quad v_3^* \quad \end{array} \\ \boldsymbol{B}^* = & \begin{bmatrix} -\frac{1}{4}(1+s)\frac{2}{L} & 0 & -\frac{1}{4}(1-s)\frac{2}{L} & 0 \\ 0 & \frac{1}{4}(1-r)\frac{2}{t} & 0 & -\frac{1}{4}(1-r)\frac{2}{t} \\ \frac{1}{4}(1-r)\frac{2}{t} & -\frac{1}{4}(1+s)\frac{2}{L} & -\frac{1}{4}(1-r)\frac{2}{t} & -\frac{1}{4}(1-s)\frac{2}{L} \end{bmatrix} \text{ etc} \end{array} \tag{19.7}$$

$$
\boldsymbol{B}_{\text{beam}} = \begin{array}{c} \\ \\ \end{array}
\begin{array}{cccc}
 & u_2 & v_2 & \theta_2 \\
\end{array}
$$

$$
\boldsymbol{B}_{\text{beam}} = \left[\begin{array}{cccc}
 & -\frac{1}{L} & 0 & \frac{t}{2L}s \\
\cdots & \emptyset & \emptyset & \emptyset \\
 & 0 & -\frac{1}{L} & -\frac{1}{2}(1-r)
\end{array}\right] \sim \left(\begin{array}{c}
\frac{\partial u}{\partial x} \\
\cancelto{0}{\frac{\partial v}{\partial y}} \\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\end{array}\right) \tag{19.8}
$$

$$v(r) = \frac{1}{2}(1-r)v_2 \tag{19.9}$$

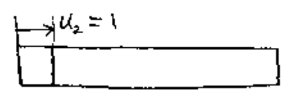
$$u(r) = \frac{1}{2}(1-r)u_2 - \frac{st}{4}(1-r)\theta_2 \tag{19.10}$$

at $r = -1$,

$$v(-1) = v_2 \tag{19.11}$$

$$u(-1) = -\frac{st}{2}\theta_2 + u_2 \tag{19.12}$$

Kinematics is



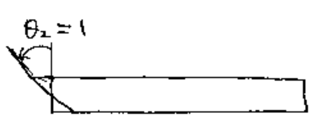
$$u(r) = \frac{1}{2}(1-r)u_2 \tag{19.13}$$

results into ϵ_{xx}

$$\rightarrow \epsilon_{xx} = \frac{\partial u}{\partial r} \cdot \frac{2}{L} = -\frac{1}{L} \tag{19.14}$$

$$u(r,s) = -\frac{st}{4}(1-r)\theta_2 \tag{19.15}$$

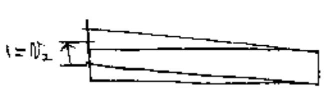
results into ϵ_{xx}, γ_{xy}



$$\rightarrow \epsilon_{xx} = \frac{st}{2L} \tag{19.16}$$

$$\gamma_{xy} = \frac{\partial u}{\partial s} \cdot \frac{2}{t} = -\frac{1}{2}(1-r) \tag{19.17}$$

$$v(r) = \frac{1}{2}(1-r)v_2 \tag{19.18}$$



results into γ_{xy}

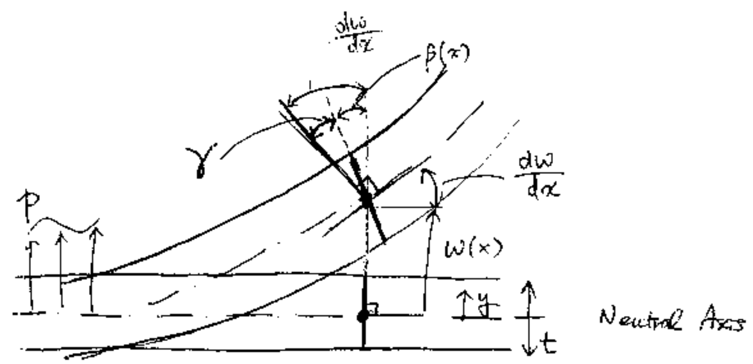
$$\rightarrow \gamma_{xy} = -\frac{1}{L} \tag{19.19}$$

For a pure bending moment, we want

$$-\frac{1}{L}v_2 - \frac{1}{2}(1-r)\theta_2 = 0 \tag{19.20}$$

for all r! $\Rightarrow$ $\boxed{\text{Impossible}}$ (except for $v_2 = \theta_2 = 0$) $\Rightarrow$ So, the element has a spurious shear strain!

Beam kinematics (Timoshenko, Reissner-Mindlin)



$$\gamma = \frac{dw}{dx} - \beta \tag{19.21}$$

$$\left(I = \frac{1}{12}bt^3\right) \tag{19.22}$$

Principle of virtual work

$$EI\int_0^L \frac{d\overline{\beta}}{dx}\frac{d\beta}{dx}dx + A_S G\int_0^L \left(\frac{d\overline{w}}{dx} - \overline{\beta}\right)\left(\frac{dw}{dx} - \beta\right)dx = \int_0^L p\overline{w}dx \tag{19.23}$$

$$A_s = kA = kbt \tag{19.24}$$

To calculate k Reading: p. 400

$$\int_A \frac{1}{2G}(\tau_a)^2 dA = \int_{A_S}\frac{1}{2G}\left(\frac{V}{A_s}\right)^2 dA_s \tag{19.25}$$

where τ_a is the actual shear stress:

$$\tau_a = \frac{3}{2}\cdot\frac{V}{A}\left[\frac{\left(\frac{t}{2}\right)^2 - y^2}{\left(\frac{t}{2}\right)^2}\right] \tag{19.26}$$

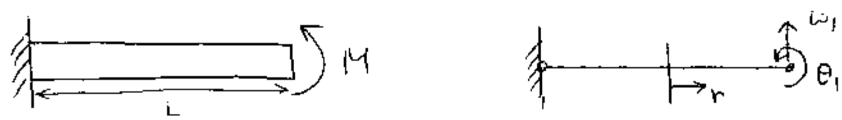
and V is the shear force. Reading: Ex. 5.23

$$\boxed{\Rightarrow k = \frac{5}{6}} \tag{19.27}$$

Now interpolate

$$w(r) = h_1 w_1 + h_2 w_2 \tag{19.28}$$
$$\beta(r) = h_1 \theta_1 + h_2 \theta_2 \tag{19.29}$$

Revisit the simple case:



$$w = \frac{1+r}{2} w_1 \tag{19.30}$$
$$\beta = \frac{1+r}{2} \theta_1 \tag{19.31}$$

Shearing strain

$$\gamma = \frac{w_1}{L} - \frac{1+r}{2} \theta_1 \tag{19.32}$$

Shear strain is not zero all along the beam. But, at $r = 0$, we can have the shear strain = 0.

$$\frac{w_1}{L} - \frac{\theta_1}{2} \text{ can be zero} \tag{19.33}$$

Namely,

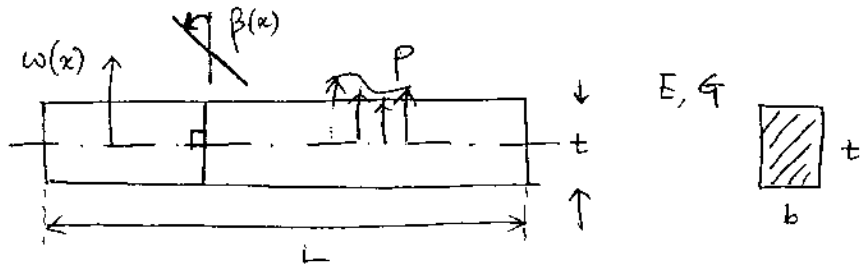
$$\frac{w_1}{L} - \frac{\theta_1}{2} = 0 \quad \text{for } \boxed{\theta_1 = \frac{2}{L} w_1} \tag{19.34}$$

2.094 — Finite Element Analysis of Solids and Fluids Fall '08

Lecture 20 - Beams, plates, and shells

Prof. K.J. Bathe MIT OpenCourseWare

Timoshenko beam theory

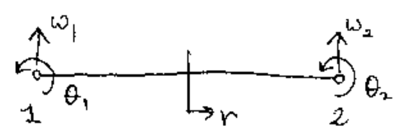


The fiber moves up and rotates and its length does not change.

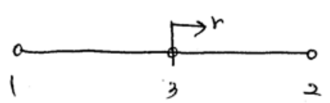
Principle of virtual displacement (Linear Analysis)

$$EI \int_0^L \left(\overline{\beta}'\right)^T \beta' dx + (Ak)G \int_0^L \left(\frac{d\overline{w}}{dx} - \overline{\beta}\right)^T \left(\frac{dw}{dx} - \beta\right) dx = \int_0^L \overline{w}^T p dx \tag{20.1}$$

Two-node element:



Three-node element:



For a q-node element,

$$\hat{\boldsymbol{u}} = \begin{bmatrix} w_1 & \cdots & w_q & \theta_1 & \cdots & \theta_q \end{bmatrix}^T \tag{20.2}$$

$$w = \boldsymbol{H}_w \hat{\boldsymbol{u}} \tag{20.3}$$

$$\beta = \boldsymbol{H}_\beta \hat{\boldsymbol{u}} \tag{20.4}$$

$$\boldsymbol{H}_w = \begin{bmatrix} h_1 & \cdots & h_q & 0 & \cdots & 0 \end{bmatrix} \tag{20.5}$$

$$\boldsymbol{H}_\beta = \begin{bmatrix} 0 & \cdots & 0 & h_1 & \cdots & h_q \end{bmatrix} \tag{20.6}$$

$$\boldsymbol{J} = \frac{dx}{dr} \tag{20.7}$$

$$\frac{dw}{dx} = \underbrace{\boldsymbol{J}^{-1}\boldsymbol{H}_{w,r}}_{\boldsymbol{B}_w}\hat{\boldsymbol{u}} \tag{20.8}$$

$$\frac{d\beta}{dx} = \underbrace{\boldsymbol{J}^{-1}\boldsymbol{H}_{\beta,r}}_{\boldsymbol{B}_\beta}\hat{\boldsymbol{u}} \tag{20.9}$$

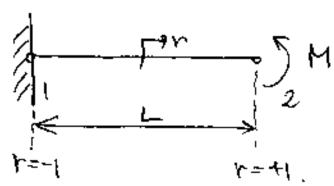
Hence we obtain

$$\left[EI\int_{-1}^{1}\boldsymbol{B}_\beta^T\boldsymbol{B}_\beta \det(\boldsymbol{J})dr + (Ak)G\int_{-1}^{1}(\boldsymbol{B}_w - \boldsymbol{H}_\beta)^T(\boldsymbol{B}_w - \boldsymbol{H}_\beta)\det(\boldsymbol{J})dr\right]\hat{\boldsymbol{u}} = \int_{-1}^{1}\boldsymbol{H}_w^T p \det(\boldsymbol{J})dr \tag{20.10}$$

$$\boxed{\boldsymbol{K}\hat{\boldsymbol{u}} = \boldsymbol{R}} \tag{20.11}$$

$\boldsymbol{K}$ is a result of the term inside the bracket in (20.10) and $\boldsymbol{R}$ is a result of the right hand side.

For the 2-node element,



$$w_1 = \theta_1 = 0 \tag{20.12}$$

$$w_2,\ \theta_2 = ? \tag{20.13}$$

$$\gamma = \frac{w_2}{L} - \frac{1+r}{2}\theta_2 \tag{20.14}$$

We cannot make γ equal to zero for every r (page 404, textbook). Because of this, we need to use about 200 elements to get an error of 10%. (Not good!)

Recall almost or fully incompressible analysis: Principle of virtual displacements:

$$\int_V \overline{\boldsymbol{\epsilon}}'^T\boldsymbol{C}'\boldsymbol{\epsilon}' dV + \int_V \overline{\epsilon}_v\,(\kappa\epsilon_v)dV = \mathcal{R} \tag{20.15}$$

u/p formulation

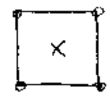
$$\int_V \overline{\boldsymbol{\epsilon}}'^T\boldsymbol{C}'\boldsymbol{\epsilon}' dV - \int_V \overline{\epsilon}_v p dV = \mathcal{R} \tag{20.16}$$

$$\int_V \overline{p}\left(\frac{p}{\kappa} + \epsilon_v\right)dV = 0 \tag{20.17}$$

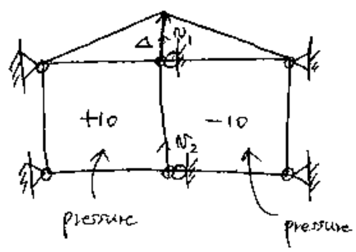
But now we needed to select wisely the interpolations of u and p. We needed to satisfy the inf-sup condition

$$\underbrace{\inf}_{q_h\in \mathrm{Q}_h}\ \underbrace{\sup}_{\boldsymbol{v}_h\in \mathrm{V}_h} \frac{\int_{\mathrm{Vol}} q_h \boldsymbol{\nabla}\cdot\boldsymbol{v}_h d\mathrm{Vol}}{\|q_h\|\|v_h\|} \geq \beta > 0 \tag{20.18}$$

4/1 element:



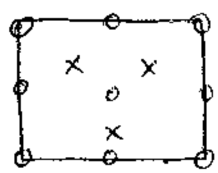
We can show mathematically that this element does not satisfy inf-sup condition. But, we can also show it by giving an example of this element which violates the inf-sup condition.



$v_1 = \triangle$, $v_2 = 0 \Rightarrow \boldsymbol{\nabla} \cdot \boldsymbol{v}_h$ for both elements is positive and the same. Now, if I choose pressures as above

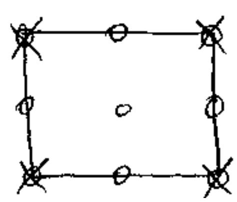
$$\int_{\text{Vol}} q_h \nabla \boldsymbol{v}_h d\text{Vol} = 0, \qquad \text{hence (20.18) is not satisfied!} \tag{20.19}$$

9/3 element



satisfies inf-sup

9/4-c



satisfies inf-sup

Getting back to beams

$$EI \int_0^L \overline{\beta}' \beta dx + (AkG) \int_0^L \left(\frac{d\overline{w}}{dx} - \overline{\beta} \right) \gamma^{AS} dx = \mathcal{R} \tag{20.20}$$

$$\int_0^L \overline{\gamma}^{AS} \left(\gamma - \gamma^{AS} \right) dx = 0 \tag{20.21}$$

where

$$\gamma = \frac{dw}{dx} - \beta, \qquad \text{from displacement interpolation} \tag{20.22}$$

$$\gamma^{AS} = \text{ Assumed shear strain interpolation} \tag{20.23}$$

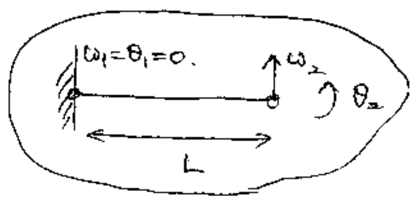
2-node element, constant shear assumption. From (20.21), Reading: Sec. 4.5.7

$$\int_0^L \left(\frac{dw}{dx} - \beta\right) \cancel{\overline{\gamma}^{AS}} dx = \int_0^L \gamma^{AS} \cancel{\overline{\gamma}^{AS}} dx \tag{20.24}$$

$$\Rightarrow -\int_{-1}^{+1} \left(\frac{1+r}{2}\theta_2\right) \cdot \frac{L}{2} dr + w_2 = \gamma^{AS} \cdot L \tag{20.25}$$

$$\Rightarrow \gamma^{AS} = \frac{w_2 - \frac{L}{2}\theta_2}{L} \tag{20.26}$$

γ^{AS} (shear strain) is equal to the displacement-based shear strain at the middle of the beam.



Use γ^{AS} in (20.20) to obtain a powerful element. For "our problem",

$$\gamma^{AS} = 0 \quad \text{hence} \quad w_2 = \frac{L}{2}\theta_2 \tag{20.27}$$

$$\Rightarrow EI \int_0^L \overline{\beta}' \beta' dx = M \left.\overline{\beta}\right|_{x=L} \tag{20.28}$$

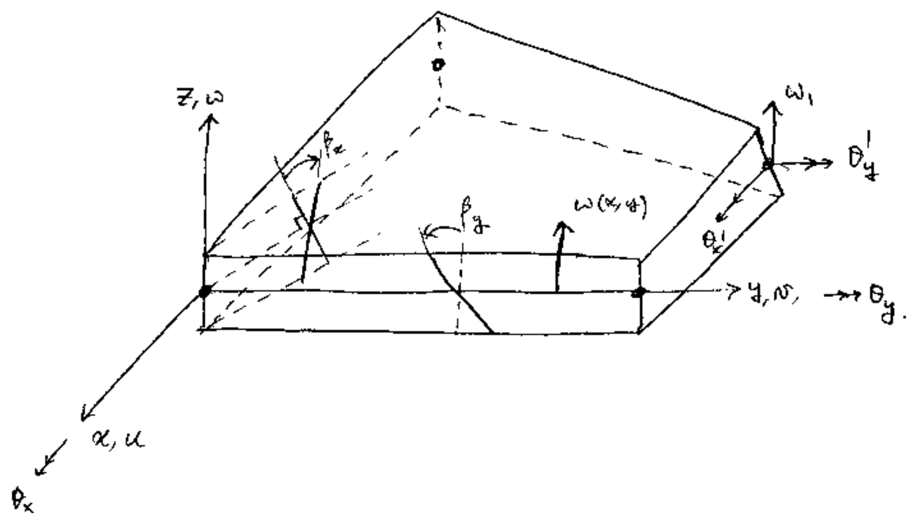
$$\Rightarrow EI \left(\left(\frac{1}{L}\right)^2 \cdot L\right) \theta_2 = M \tag{20.29}$$

$$\Rightarrow \boxed{\theta_2 = \frac{ML}{EI}, \qquad w_2 = \frac{ML^2}{2EI}} \tag{20.30}$$

(exact solutions)

Plates

Reading: Fig. 5.25, p. 421



$$\begin{cases} w = w(x, y) & \text{is the transverse displacement of the mid-surface} \\ v = -z\beta_y(x, y) \\ u = -z\beta_x(x, y) \end{cases} \tag{20.31}$$

For any particle in the plate with coordinates (x, y, z), the expressions in (20.31) hold!

We use

$$w = \sum_{i=1}^{q} h_i w_i \tag{20.32}$$

$$\beta_x = -\sum_{i=1}^{q} h_i \theta_y^i \tag{20.33}$$

$$\beta_y = +\sum_{i=1}^{q} h_i \theta_x^i \tag{20.34}$$

where q equals the number of nodes. Then the element locks in the same way as the displacement-based beam element.

2.094 — Finite Element Analysis of Solids and Fluids Fall ‘08

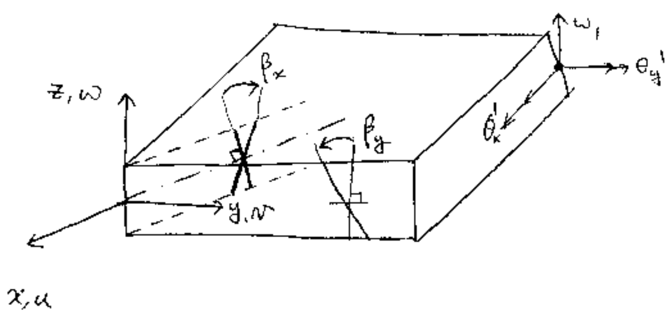
Lecture 21 - Plates and shells

Prof. K.J. Bathe MIT OpenCourseWare

Timoshenko beam theory, and Reissner-Mindlin plate theory

For plates, and shells, w, β_x, and β_y as independent variables.

w = displacement of mid-surface, $w(x,y)$



A = area of mid-surface

p = load per unit area on mid-surface

$$w = w(x,y) \tag{21.1}$$

$$w(x,y,z) = w(x,y) \tag{21.2}$$

The material particles at “any z” move in the z-direction as the mid-surface.

$$u(x,y,z) = -\beta_x z = -\beta_x(x,y)z \tag{21.3}$$
$$v(x,y,z) = -\beta_y z = -\beta_y(x,y)z \tag{21.4}$$

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial \beta_x}{\partial x} \tag{21.5}$$
$$\epsilon_{yy} = \frac{\partial v}{\partial y} = -z\frac{\partial \beta_y}{\partial y} \tag{21.6}$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -z\left(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x}\right) \tag{21.7}$$

$$\begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = -z \underbrace{\begin{pmatrix} \frac{\partial \beta_x}{\partial x} \\ \frac{\partial \beta_y}{\partial y} \\ \frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x} \end{pmatrix}}_{\boldsymbol{\kappa}} \tag{21.8}$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} - \beta_x \tag{21.9}$$
$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \frac{\partial w}{\partial y} - \beta_y \tag{21.10}$$

$$\begin{pmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{pmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \boldsymbol{C} \cdot \boldsymbol{\epsilon} \tag{21.11}$$

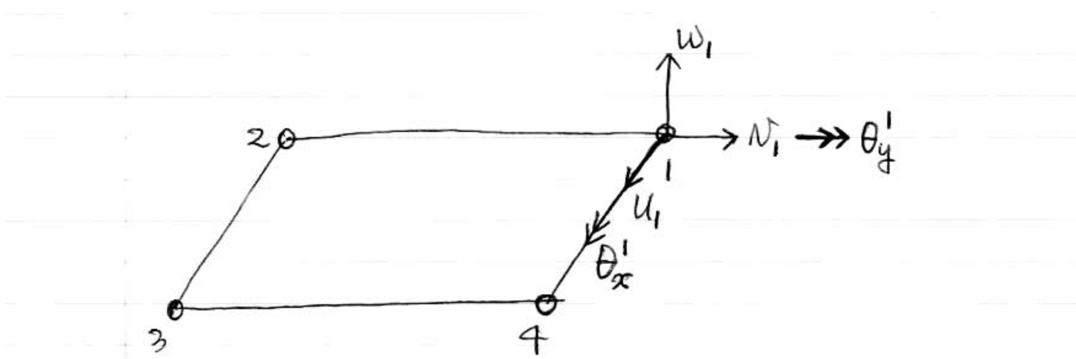
(plane stress)

$$\begin{pmatrix} \tau_{xz} \\ \tau_{yz} \end{pmatrix} = \frac{E}{2(1+\nu)} \begin{pmatrix} \gamma_{xz} \\ \gamma_{yz} \end{pmatrix} = G\boldsymbol{\gamma} \tag{21.12}$$

Principle of virtual work for the plate:

$$\begin{aligned} \int_A \int_{-\frac{t}{2}}^{+\frac{t}{2}} \begin{pmatrix} \overline{\epsilon}_{xx} & \overline{\epsilon}_{yy} & \overline{\gamma}_{xy} \end{pmatrix} \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{pmatrix} dz dA + \\ k \int_A \int_{-\frac{t}{2}}^{+\frac{t}{2}} \begin{pmatrix} \overline{\gamma}_{xz} & \overline{\gamma}_{yz} \end{pmatrix} G \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \gamma_{xz} \\ \gamma_{yz} \end{pmatrix} dz dA = \int_A \overline{w} p dA \end{aligned} \tag{21.13}$$

Consider a flat element:



$$\Rightarrow \boldsymbol{K}_b \begin{pmatrix} w_1 \\ \theta_x^1 \\ \theta_y^1 \\ \vdots \end{pmatrix}, \text{ also } \boldsymbol{K}_{\text{pl. str.}} \begin{pmatrix} u_1 \\ v_1 \\ \vdots \end{pmatrix} \tag{21.14}$$

where $\boldsymbol{K}_b$ is 12x12 and $\boldsymbol{K}_{\text{pl. str.}}$ is 8x8.

For a flat element:

$$\Rightarrow \begin{bmatrix} \boldsymbol{K}_b & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{K}_{\text{pl. str.}} \end{bmatrix} \begin{pmatrix} w_1 \\ \theta_x^1 \\ \theta_y^1 \\ \vdots \\ \theta_y^4 \\ u_1 \\ v_1 \\ \vdots \\ v_4 \end{pmatrix} = \cdots \tag{21.15}$$

$$u = \sum h_i u_i \tag{21.16}$$

$$v = \sum h_i v_i \tag{21.17}$$

$$w = \sum h_i w_i \tag{21.18}$$

$$\beta_x = -\sum h_i \theta_y^i \tag{21.19}$$

$$\beta_y = \sum h_i \theta_x^i \tag{21.20}$$

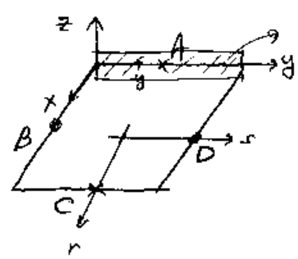
From (21.13)

$$\int_A \overline{\boldsymbol{\kappa}}^T \frac{Et^3}{12\left(1-\nu^2\right)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \boldsymbol{\kappa} dA + \int_A \overline{\boldsymbol{\gamma}}^T Gt \cdot k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{\gamma} dA \tag{21.21}$$

where k is the shear correction factor.

Next, evaluate $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$, $\frac{\partial \beta_x}{\partial x}$, ...etc. $\Rightarrow$ $\boxed{\text{This element, as it is, locks!}}$

This displacement-based element "locks in shear". We need to change the transverse shear interpolations.



consider this section as a beam.

$$\gamma_{yz} = \frac{1}{2}(1-r)\gamma_{yz}^A + \frac{1}{2}(1+r)\gamma_{yz}^C \tag{21.22}$$

where

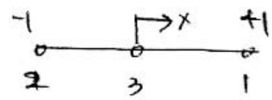
$$\gamma_{yz}^A = \left(\frac{\partial w}{\partial y} - \beta_y\right)\Bigg|_{\text{evaluated at A}} \tag{21.23}$$

from the w, β_y displacement interpolations.

$$\gamma_{xz} = \frac{1}{2}(1-s)\gamma_{xz}^B + \frac{1}{2}(1+s)\gamma_{xz}^D \tag{21.24}$$

with this mixed interpolation, the element works. Called MITC interpolation (for **m**ixed **i**nterpolated-**t**ensional **c**omponents)

Aside: Why not just neglect transverse shears, as in Kirchhoff plate theory?

- If we do, $\gamma_{xz} = \frac{\partial w}{\partial x} - \beta_x = 0 \Rightarrow \beta_x = \frac{\partial w}{\partial x}$
- Therefore we have $\left(\frac{\partial^2 w}{\partial x^2}, \cdots\right)$ in strains, so we need continuity also for $\left(\frac{\partial w}{\partial x} \cdots\right)$
- 

$$w = \frac{1}{2}r(1+r)w_1 - \frac{1}{2}r(1-r)w_2 + \left(1-r^2\right)w_3$$

w_2 and w_3 never affect w_1 ($\therefore w|_{r=1} = w_1$).

But,

$$\frac{\partial w}{\partial r} = \frac{1}{2}(1+2r)w_1 - \frac{1}{2}(1-2r)w_2 - 2rw_3$$

w_2 and w_3 affect $\left.\frac{\partial w}{\partial r}\right|_1$. $\Rightarrow$ This results in difficulties to develop a good element based on Kirchhoff theory.

With Reissner-Mindlin theory, we independently interpolate rotations such that this problem does not arise.

For flat structures, we can superimpose the plate bending and plane stress element stiffness. For shells, curved structures, we need to develop/use curved elements, see references.

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