2.094 — Finite Element Analysis of Solids and Fluids Fall '08

Lecture 12 - Total Lagrangian formulation

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We discussed:

$$ {}_0^t\boldsymbol{X} = \left[\frac{\partial^t x_i}{\partial^0 x_j}\right] \qquad \Rightarrow \quad d^t\boldsymbol{x} = {}_0^t\boldsymbol{X}\, d^0\boldsymbol{x}, \quad d^0\boldsymbol{x} = \left({}_0^t\boldsymbol{X}\right)^{-1} d^t\boldsymbol{x} \tag{12.1} $$

$$ {}_0^t\boldsymbol{C} = {}_0^t\boldsymbol{X}^T\, {}_0^t\boldsymbol{X} \tag{12.2} $$

$$ d^0\boldsymbol{x} = {}_t^0\boldsymbol{X}\, d^t\boldsymbol{x} \qquad \text{where } {}_t^0\boldsymbol{X} = \left({}_0^t\boldsymbol{X}\right)^{-1} = \left[\frac{\partial^0 x_i}{\partial^t x_j}\right] \tag{12.3} $$

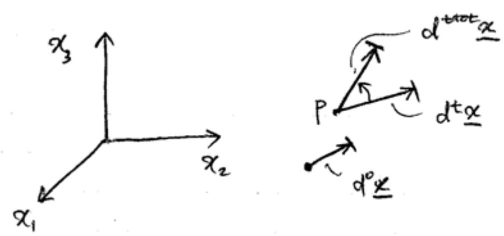
The Green-Lagrange strain:

$$ {}_0^t\boldsymbol{\epsilon} = \frac{1}{2}\left({}_0^t\boldsymbol{X}^T\, {}_0^t\boldsymbol{X} - \boldsymbol{I}\right) = \frac{1}{2}\left({}_0^t\boldsymbol{C} - \boldsymbol{I}\right) \tag{12.4} $$

Polar decomposition:

$$ {}_0^t\boldsymbol{X} = {}_0^t\boldsymbol{R}\,{}_0^t\boldsymbol{U} \Rightarrow {}_0^t\boldsymbol{\epsilon} = \frac{1}{2}\left(\left({}_0^t\boldsymbol{U}\right)^2 - \boldsymbol{I}\right) \tag{12.5} $$

We see, physically that:



where $d^{t+\Delta t}\boldsymbol{x}$ and $d^t\boldsymbol{x}$ are the same lengths
$\Rightarrow$ the components of the G-L strain do not change.

Note in FEA

$$ \left.\begin{aligned} {}^0x_i &= \sum_k h_k\, {}^0x_i^k \\ {}^tu_i &= \sum_k h_k\, {}^tu_i^k \end{aligned}\right\} \quad \text{for an element} \tag{12.6} $$

$$ {}^tx_i = {}^0x_i + {}^tu_i \rightarrow \quad \text{for any particle} \tag{12.7} $$

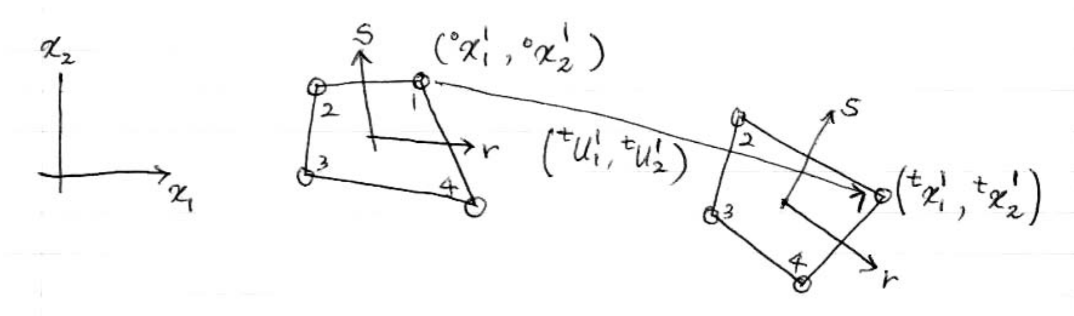
Hence for the element

$$ {}^tx_i = \sum_k h_k\, {}^0x_i^k + \sum_k h_k\, {}^tu_i^k \tag{12.8} $$

$$ = \sum_k h_k \left({}^0x_i^k + {}^tu_i^k\right) \tag{12.9} $$

$$ = \sum_k h_k\, {}^tx_k^k \tag{12.10} $$

E.g., $k = 4$



2nd Piola-Kirchhoff stress

$$ {}_0^t\boldsymbol{S} = \frac{{}^0\rho}{{}^t\rho}\, {}_t^0\boldsymbol{X}\, {}^t\boldsymbol{\tau}\, {}_t^0\boldsymbol{X}^T \rightarrow \quad \text{components also independent of a rigid body rotation} \tag{12.11} $$

Then

$$ \int_{{}^0V} {}_0^tS_{ij}\, \delta\, {}_0^t\epsilon_{ij}\, d^0V = \int_{{}^tV} {}^t\tau_{ij}\, \delta\, {}_te_{ij}\, d^tV = {}^t\mathcal{R} \tag{12.12} $$

We can use an incremental decomposition of stress/strain.

$$ {}_0^{t+\Delta t}\boldsymbol{S} = {}_0^t\boldsymbol{S} + {}_0\boldsymbol{S} \tag{12.13} $$
$$ {}_0^{t+\Delta t}S_{ij} = {}_0^tS_{ij} + {}_0S_{ij} \tag{12.14} $$
$$ {}_0^{t+\Delta t}\boldsymbol{\epsilon} = {}_0^t\boldsymbol{\epsilon} + {}_0\boldsymbol{\epsilon} \tag{12.15} $$
$$ {}_0^{t+\Delta t}\epsilon_{ij} = {}_0^t\epsilon_{ij} + {}_0\epsilon_{ij} \tag{12.16} $$

Assume the solution is kown at time t, calculate the solution at time $t+\Delta t$. Hence, we apply (12.12) at time $t + \Delta t$:

$$ \int_{{}^0V} {}_0^{t+\Delta t}S_{ij}\, \delta\, {}_0^{t+\Delta t}\epsilon_{ij}\, d^0V = {}^{t+\Delta t}\mathcal{R} \tag{12.17} $$

Look at $\delta\, {}_0^t\epsilon_{ij}$:

$$ \delta\, {}_0^t\epsilon_{ij} = \delta \frac{1}{2}\left({}_0^tu_{i,j} + {}_0^tu_{j,i} + {}_0^tu_{k,i}\, {}_0^tu_{k,j} \right) \tag{12.18a} $$
$$ \delta\, {}_0^t\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial^0 x_j} + \frac{\partial \delta u_j}{\partial^0 x_i} + \frac{\partial \delta u_k}{\partial^0 x_i} \cdot \frac{\partial^t u_k}{\partial^0 x_j} + \frac{\partial^t u_k}{\partial^0 x_i} \cdot \frac{\partial \delta u_k}{\partial^0 x_j} \right) \tag{12.18b} $$
$$ \delta\, {}_0^t\epsilon_{ij} = \frac{1}{2}\left(\delta\, {}_0u_{i,j} + \delta\, {}_0u_{j,i} + \delta\, {}_0u_{k,i}\, {}_0^tu_{k,j} + {}_0^tu_{k,i}\, \delta\, {}_0u_{k,j} \right) \tag{12.18c} $$

We have

$$ {}_0^{t+\Delta t}\epsilon_{ij} - {}_0^t\epsilon_{ij} = {}_0\epsilon_{ij} \tag{12.19} $$
$$ {}_0\epsilon_{ij} = {}_0e_{ij} + {}_0\eta_{ij} \tag{12.20} $$

where ${}_0e_{ij}$ is the linear incremental strain, ${}_0\eta_{ij}$ is the nonlinear incremental strain, and

$$ {}_0e_{ij} = \frac{1}{2}\left({}_0u_{i,j} + {}_0u_{j,i} + \underbrace{{}^t_0u_{k,i}\,{}_0u_{k,j} + {}_0u_{k,i}\,{}^t_0u_{k,j}}_{\text{initial displ. effect}} \right) \tag{12.21} $$

$$ {}_0\eta_{ij} = \frac{1}{2}\,{}_0u_{k,i}\,{}_0u_{k,j} \tag{12.22} $$

where

$$ {}_0u_{k,j} = \frac{\partial u_k}{\partial {}^0x_j}, \qquad \boxed{u_k = {}^{t+\Delta t}u_k - {}^tu_k} \tag{12.23} $$

Note

$$ \delta\,{}^{t+\Delta t}\epsilon_{ij} = \delta\,{}_0\epsilon_{ij} \qquad \left(\because\ \delta\,{}^t_0\epsilon_{ij} = 0 \text{ when changing the configuration at } t+\Delta t\right) \tag{12.24} $$

From (12.17):

$$ \int_{{}^0V} \left({}^t_0S_{ij} + {}_0S_{ij}\right)\left(\delta_0e_{ij} + \delta_0\eta_{ij}\right) d^0V $$

$$ = \int_{{}^0V} \left({}^t_0S_{ij}\delta_0e_{ij} + {}_0S_{ij}\delta_0e_{ij} + {}^t_0S_{ij}\delta_0\eta_{ij} + {}_0S_{ij}\delta_0\eta_{ij}\right) d^0V \tag{12.25} $$

$$ = {}^{t+\Delta t}\mathcal{R} \tag{12.26} $$

Linearization

$$ \int_{{}^0V} \left(\overbrace{{}_0S_{ij}\delta_0e_{ij}}^{{}^t_0\boldsymbol{K}_L\boldsymbol{U}} + \overbrace{{}^t_0S_{ij}\delta_0\eta_{ij}}^{{}^t_0\boldsymbol{K}_{NL}\boldsymbol{U}} \right) d^0V = {}^{t+\Delta t}\mathcal{R} - \underbrace{\int_{{}^0V} {}^t_0S_{ij}\delta_0e_{ij}\,d^0V}_{{}^t_0\boldsymbol{F}} \tag{12.27} $$

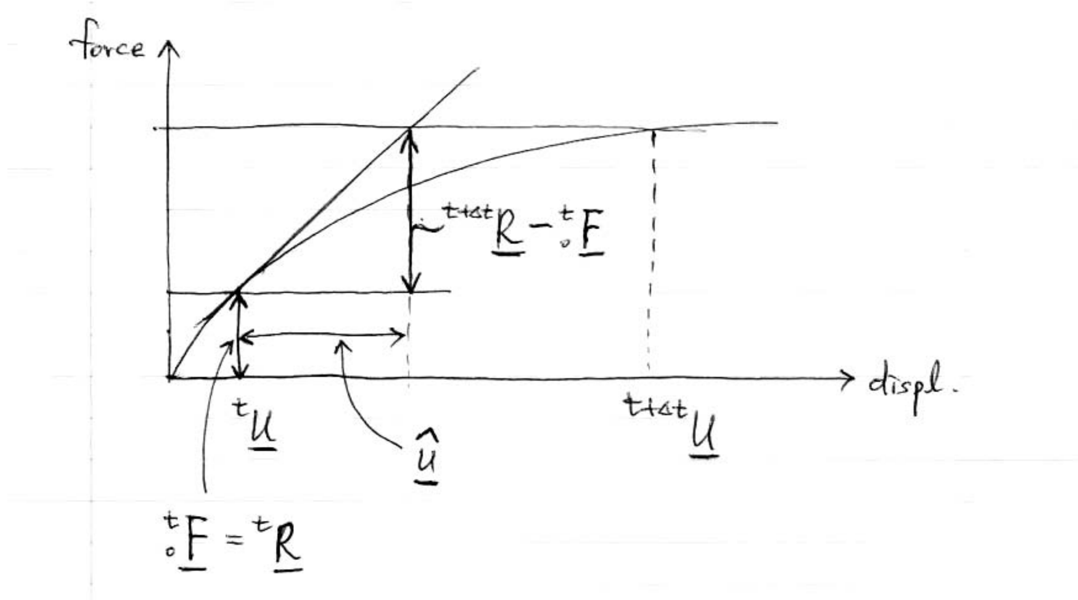
We use,

$$ {}_0S_{ij} \simeq {}_0C_{ijrs}\,{}_0e_{rs} \tag{12.28} $$

We arrive at, with the finite element interpolations,

$$ \left({}^t_0\boldsymbol{K}_L + {}^t_0\boldsymbol{K}_{NL}\right)\boldsymbol{U} = {}^{t+\Delta t}\boldsymbol{R} - {}^t_0\boldsymbol{F} \tag{12.29} $$

where $\boldsymbol{U}$ is the nodal displacement increment.



Left hand side as before but using $(k-1)$ and right hand side is

$$= {}^{t+\Delta t}\mathcal{R} - \int_{{}^0V} {}^{t+\Delta t}_{\;\;0}S_{ij}\,\delta\,{}^{t+\Delta t}_{\;\;0}\epsilon_{ij}^{(k-1)}\,d^0V \tag{12.30}$$

gives

$${}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}_{\;\;0}\boldsymbol{F}^{(k-1)} \tag{12.31}$$

In the full N-R iteration, we use

$$\left({}^{t+\Delta t}_{\;\;0}\boldsymbol{K}_L^{(k-1)} + {}^{t+\Delta t}_{\;\;0}\boldsymbol{K}_{NL}^{(k-1)}\right)\Delta\boldsymbol{U}^{(k)} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}_{\;\;0}\boldsymbol{F}^{(k-1)} \tag{12.32}$$