2.161 Signal Processing: Continuous and Discrete
Fall 2008

Massachusetts Institute of Technology
Department of Mechanical Engineering

2.161 *Signal Processing - Continuous and Discrete*
Fall Term 2008

Lecture 3[1]

Reading:

- Class handout: Frequency Domain Methods

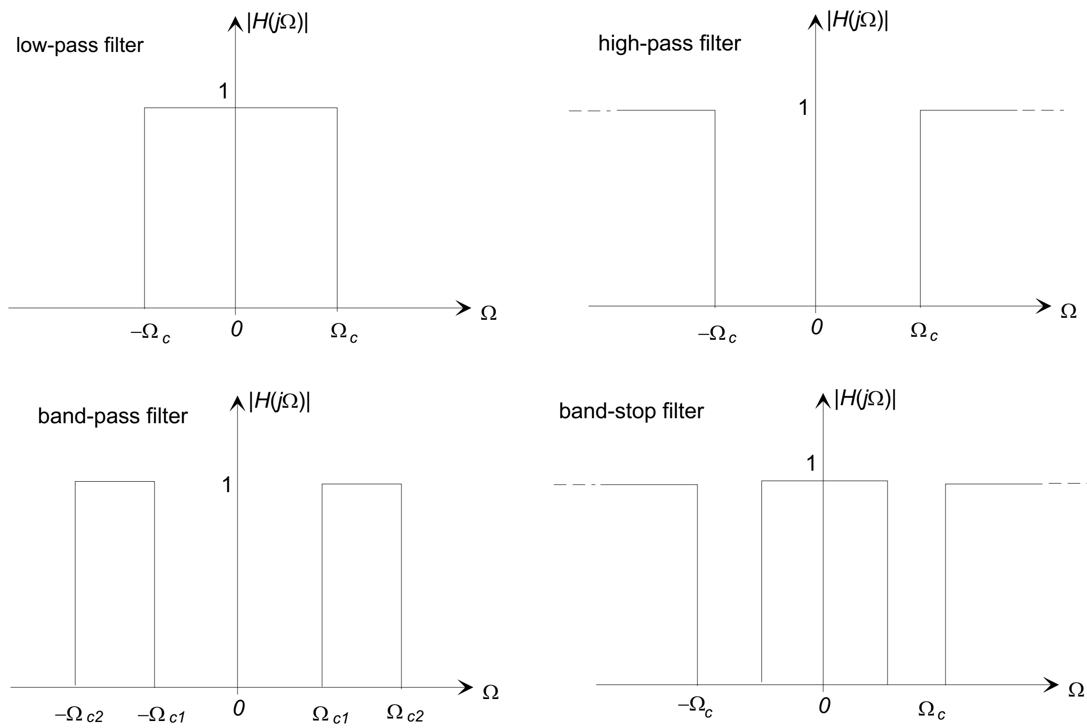
1 The Fourier Series and Transform

In Lecture 2 we looked at the response of LTI continuous systems to sinusoidal inputs of the form

$$u(t) = A\sin(\Omega t + \phi)$$

and saw that the system was characterized by the *frequency response function* $H(j\Omega)$.

In signal processing work, linear filters are usually specified by a desired frequency response function. (We will see that often the *magnitude function* $|H(j\omega)|$ alone is used to specify a filter). The following figure shows the four basic forms of *ideal* linear filters:



In this lecture we generalize the response of LTI systems to non-sinusoidal inputs. We do this using Fourier methods.

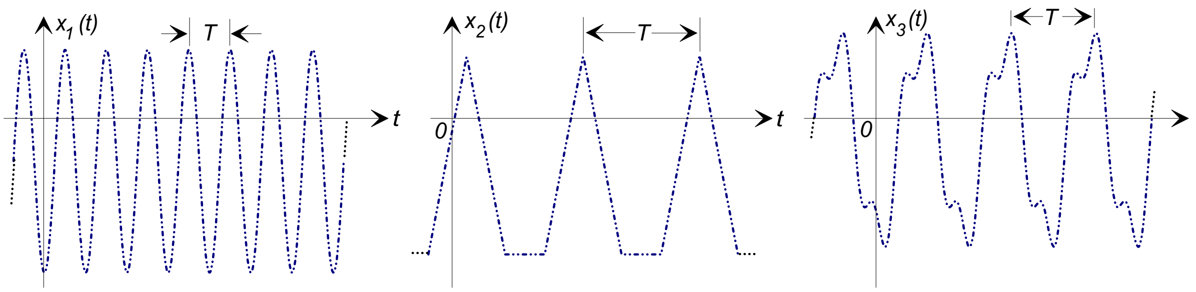
[1]

2 Periodic Input Functions - The Fourier Series

In general, a *periodic function* is a function that satisfies the relationship:

$$x(t) = x(t + T)$$

for all t, or $x(t) = x(t + nT)$ for $n = \pm1, \pm2, \pm3, \ldots$, where T is defined as the *period*. Some examples of periodic functions is shown below.



The *fundamental angular frequency* Ω_0 (in radians/second) of a periodic waveform is defined directly from the period

$$\Omega_0 = \frac{2\pi}{T},$$

and the *fundamental frequency* F_0 (in Hz) is

$$F_0 = \frac{1}{T}$$

so that $\Omega_0 = 2\pi F_0$.

- Any periodic function with period T is also be periodic at intervals of nT for any positive integer n. Similarly any waveform with a period of T/n is periodic at intervals of T seconds.
- Two waveforms whose periods, or frequencies, are related by a simple integer ratio are said to be *harmonically related*.
- If two harmonically related functions are summed together to produce a new function $g(t) = x_1(t) + x_2(t)$, then $g(t)$ will be periodic with a period defined by the longest period of the two components. In general, when harmonically related waveforms are added together the resulting function is also periodic with a repetition period equal to the fundamental period.

■ Example 1

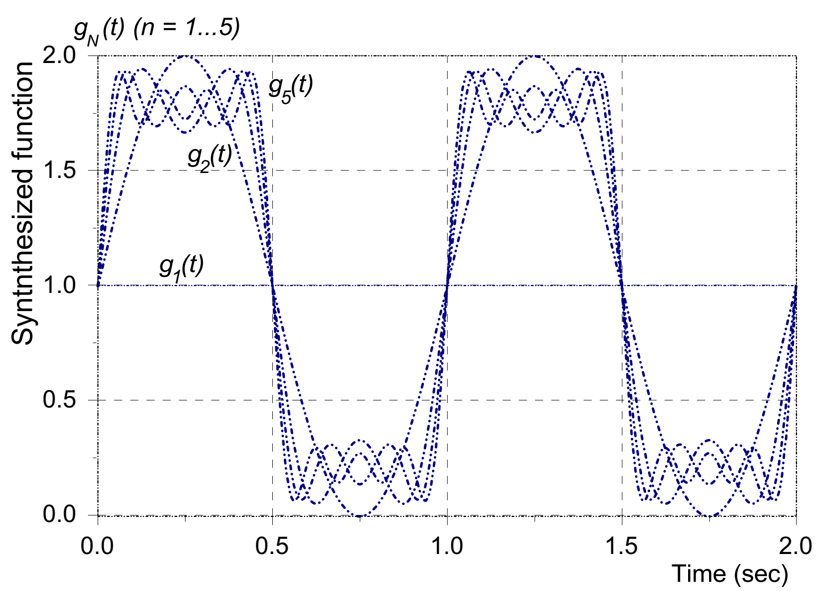
A family of waveforms $g_N(t)$ ($N = 1, 2 \ldots 5$) is formed by adding together the first N of up to five component functions, that is

$$g_N(t) = \sum_{n=1}^{N} x_n(t) \qquad\qquad 1 < N \leq 5$$

where

$$\begin{aligned} x_1(t) &= 1 \\ x_2(t) &= sin(2\pi t) \\ x_3(t) &= \frac{1}{3} sin(6\pi t) \\ x_4(t) &= \frac{1}{5} sin(10\pi t) \\ x_5(t) &= \frac{1}{7} sin(14\pi t). \end{aligned}$$

The first term is a constant, and the four sinusoidal components are harmonically related, with a fundamental frequency of $\Omega_0 = 2\pi$ rad/s and a fundamental period of $T = 2\pi/\Omega_0 = 1$ second. (The constant term may be considered to be periodic with any arbitrary period, but is commonly considered to have a frequency of zero rad/s.) The figure below shows the evolution of the function that is formed as more of the individual terms are included into the summation. Notice that in all cases the period of the resulting $g_N(t)$ remains constant and equal to the period of the fundamental component (1 second). In this particular case, it can be seen that the sum is tending toward a square wave as more terms are included.



The Fourier series representation of an arbitrary periodic waveform $x(t)$ (subject to some general conditions described later) is as an infinite sum of harmonically related sinusoidal components, commonly written in the following three equivalent forms

$$\begin{aligned} x(t) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\Omega_0 t) + b_n \sin(n\Omega_0 t)\right) & (1) \\ &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \mathcal{A}_n \sin(n\Omega_0 t + \phi_n) & (2) \\ &= \sum_{n=-\infty}^{+\infty} X_n e^{jn\Omega_0 t}. & (3) \end{aligned}$$

In each representation knowledge of the fundamental frequency Ω_0, and the sets of Fourier coefficients $\{a_n\}$ and $\{b_n\}$ ($n = 0 \ldots \infty$), or $\{\mathcal{A}_n\}$ and $\{\phi_n\}$) ($n = 0 \ldots \infty$), or $\{X_n\}$ ($n = -\infty \ldots \infty$) is sufficient to completely define the waveform $x(t)$.

These representations are related by

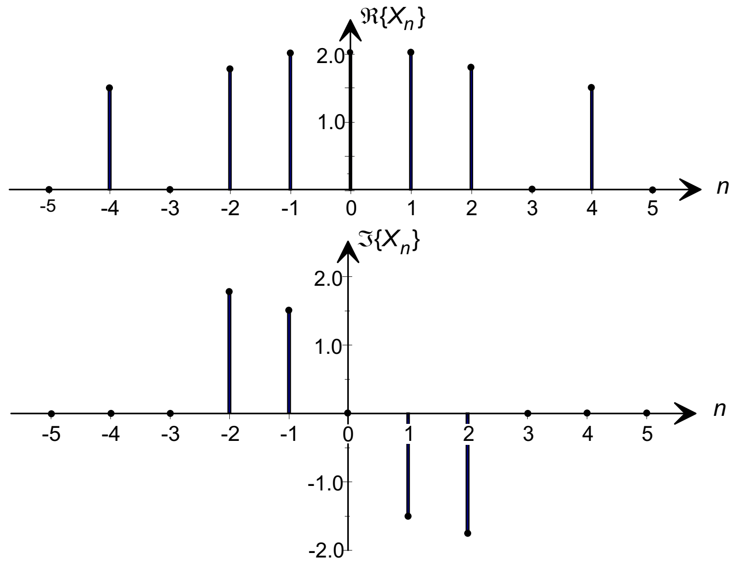
$$\begin{aligned} \mathcal{A}_n &= \sqrt{a_n^2 + b_n^2} \\ \phi_n &= \tan^{-1}(a_n/b_n). \end{aligned}$$

and

$$\begin{aligned} X_n &= 1/2(a_n - jb_n) \\ X_{-n} &= 1/2(a_n + jb_n) \end{aligned}$$

See the class handout for details.

The *spectrum* of a periodic waveform is the set of all of Fourier coefficients in any of the representations, for example $\{\mathcal{A}_n\}$ and $\{\phi_n\}$, expressed as a function of frequency. Because the harmonic components exist at discrete frequencies, periodic functions are said to exhibit *line (or discrete) spectra*, and it is common to express the spectrum graphically with frequency Ω as the independent axis, and with the Fourier coefficients plotted as lines at intervals of Ω_0. The first two forms of the Fourier series, based upon Eqs. (**??**) and (**??**), generate "one-sided" spectra because they are defined from positive values of n only, whereas the complex form defined by Eq. (**??**) generates a "two-sided" spectrum because its summation requires positive and negative values of n. The figure below shows a complex spectrum (Eq. **??**).



2.1 Computation of the Fourier Coefficients

Section (2.1) of the class handout derives the finite (truncated) Fourier series as a least-squares approximation to the periodic function $x(t)$. The results show that for the complex representation

$$X_n = \frac{1}{T}\int_{t_0}^{t_0+T} x_p(t)e^{-jn\Omega_0 t}dt,$$

and for the real representation

$$\begin{aligned} a_n &= \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(n\Omega_0 t)dt \\ b_n &= \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(n\Omega_0 t)dt. \end{aligned}$$

The results are summarized in the following table:

	Sinusoidal formulation	Exponential formulation
Synthesis:	$x(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(n\Omega_0 t) + b_n\sin(n\Omega_0 t)\right)$	$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{jn\Omega_0 t}$
Analysis:	$a_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\cos(n\Omega_0 t)dt$ $b_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)\sin(n\Omega_0 t)dt$	$X_n = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)e^{-jn\Omega_0 t}dt$

2.2 Properties of the Fourier Series

The following are some of the important properties of the Fourier series:

(1) **Existence of the Fourier Series** For the series to exist, the Fourier analysis integral must converge. A set of three sufficient conditions, known as the Dirichelet conditions, guarantee the existence of a Fourier series for a given periodic waveform $x(t)$. They are

- The function x(t) must be *absolutely integrable* over any period, that is

$$\int_{t_0}^{t_0+T} |x(t)|\, dt < \infty$$

  for any t_0.

- There must be at most a finite number of maxima and minima in the function $x(t)$ within any period.
- There must be at most a finite number of discontinuities in the function $x(t)$ within any period, and all such discontinuities must be finite in magnitude.

These requirements are satisfied by almost all waveforms found in engineering practice. The Dirichelet conditions are a *sufficient* set of conditions to guarantee the existence of a Fourier series representation. They are not *necessary* conditions, and there are some functions that have a Fourier series representation without satisfying all three conditions.

(2) **Linearity of the Fourier Series Representation** The Fourier analysis and synthesis operations are linear. Consider two periodic functions $g(t)$ and $h(t)$ with identical periods T, and their complex Fourier coefficients

$$\begin{aligned} G_n &= \frac{1}{T}\int_0^T g(t)e^{-jn\Omega_0 t}dt \\ H_n &= \frac{1}{T}\int_0^T h(t)e^{-jn\Omega_0 t}dt \end{aligned}$$

and a third function defined as a weighted sum of $g(t)$ and $h(t)$

$$x(t) = ag(t) + bh(t)$$

where a and b are constants. The linearity property, which may be shown by direct substitution into the integral, states that the Fourier coefficients of $x(t)$ are

$$X_n = aG_n + bH_n,$$

that is the Fourier series of a weighted sum of two time-domain functions is the weighted sum of the individual series.

(3) Even and Odd Functions If $x(t)$ exhibits symmetry about the $t = 0$ axis the Fourier series representation may be simplified. If $x(t)$ is an even function of time, that is $x(-t) = x(t)$, the complex Fourier series has coefficients X_n that are purely real, with the result that the real series contains only cosine terms, so that Eq. (**??**) simplifies to

$$x(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\Omega_0 t).$$

Similarly if $x(t)$ is an odd function of time, that is $x(-t) = -x(t)$, the coefficients X_n are imaginary, and the one-sided series consists of only sine terms:

$$x(t) = \sum_{n=1}^{\infty} b_n \sin(n\Omega_0 t).$$

Notice that an odd function requires that $x(t)$ have a zero average value.

(4) The Fourier Series of a Time Shifted Function If the periodic function $x(t)$ has a Fourier series with complex coefficients X_n, the series representing a "time-shifted" version $g(t) = x(t + \tau)$ has coefficients $e^{-jn\Omega_0\tau}X_n$. If

$$X_n = \frac{1}{T}\int_0^T x(t)e^{-jn\Omega_0 t}dt$$

then

$$G_n = \frac{1}{T}\int_0^T f(t+\tau)e^{-jn\Omega_0 t}dt.$$

Changing the variable of integration $\nu = t + \tau$ gives

$$\begin{aligned} G_n &= \frac{1}{T}\int_\tau^{\tau+T} f(\nu)e^{-jn\Omega_0(\nu-\tau)}d\nu \\ &= e^{jn\Omega_0\tau}\frac{1}{T}\int_\tau^{\tau+T} f(\nu)e^{-jn\Omega_0\nu t}d\nu \\ &= e^{jn\Omega_0\tau}X_n. \end{aligned}$$

If the nth spectral component is written in terms of its magnitude and phase

$$x_n(t) = \mathcal{A}_n \sin(n\Omega_0 t + \phi_n)$$

then

$$\begin{aligned} x_n(t+\tau) &= \mathcal{A}_n \sin\left(n\Omega_0(t+\tau) + \phi_n\right) \\ &= \mathcal{A}_n \sin\left(n\Omega_0 t + \phi_n + n\Omega_0\tau\right). \end{aligned}$$

The additional phase shift $n\Omega_0\tau$, caused by the time shift τ, is directly proportional to the frequency of the component $n\Omega_0$.

(5) Interpretation of the Zero Frequency Term The coefficients X_0 in the complex series and a_0 in the real series are somewhat different from all of the other terms for they correspond to a harmonic component with zero frequency. The complex analysis equation shows that

$$X_0 = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)dt$$

and the real analysis equation gives

$$\frac{1}{2}a_0 = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)dt$$

which are both simply the average value of the function over one complete period.

If a function $x(t)$ is modified by adding a constant value to it, the only change in its series representation is in the coefficient of the zero-frequency term, either X_0 or a_0.

2.3 The Response of Linear Systems to Periodic Inputs

Consider a linear single-input, single-output system with a frequency response function $H(j\Omega)$. Let the input $u(t)$ be a periodic function with period T, and write it in terms of a real Fourier series:

$$u(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \mathcal{A}_n \sin(n\Omega_0 t + \phi_n)$$

The nth real harmonic input component, $u_n(t) = \mathcal{A}_n \sin(n\Omega_0 t + \phi_n)$, generates an output sinusoidal component $y_n(t)$ with a magnitude and a phase that is determined by the system's frequency response function $H(j\Omega)$:

$$y_n(t) = |H(jn\Omega_0)|\, \mathcal{A}_n \sin(n\Omega_0 t + \phi_n + \angle H(jn\Omega_0)).$$

The principle of superposition states that the total output $y(t)$ is the sum of all such component outputs, or

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} y_n(t) \\ &= \frac{1}{2}a_0 H(j0) + \sum_{n=1}^{\infty} \mathcal{A}_n \left|H(jn\Omega_0)\right| \sin\left(n\Omega_0 t + \phi_n + \angle H(jn\Omega_0)\right), \end{aligned}$$

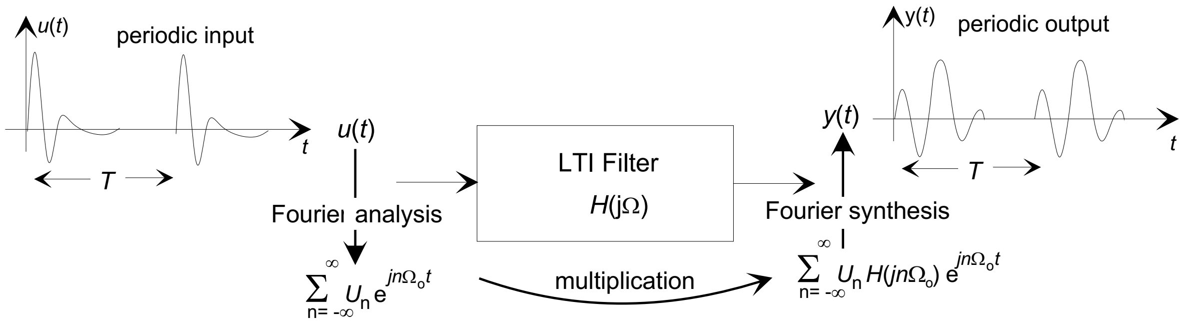
which is itself a Fourier series with the same fundamental and harmonic frequencies as the input. The output $y(t)$ is therefore also a periodic function with the same period T as the input, but because the system frequency response function has modified the relative magnitudes and the phases of the components, the waveform of the output $y(t)$ differs in form and appearance from the input $u(t)$.

In the complex formulation the input waveform is decomposed into a set of complex exponentials $u_n(t) = U_n e^{jn\Omega_0 t}$. Each such component is modified by the system frequency response so that the output component is

$$y_n(t) = H(jn\Omega_0) U_n e^{jn\Omega_0 t}$$

and the complete output Fourier series is

$$y(t) = \sum_{n=-\infty}^{\infty} y_n(t) = \sum_{n=-\infty}^{\infty} H(jn\Omega_0)U_n e^{jn\Omega_0 t}.$$



The system $H(j\Omega)$ acts as a frequency-domain filter and modifies the input waveform $u(t)$ by (1) selectively amplifying/attenuating the spectral components, and (2) applying a frequency dependent phase shift.

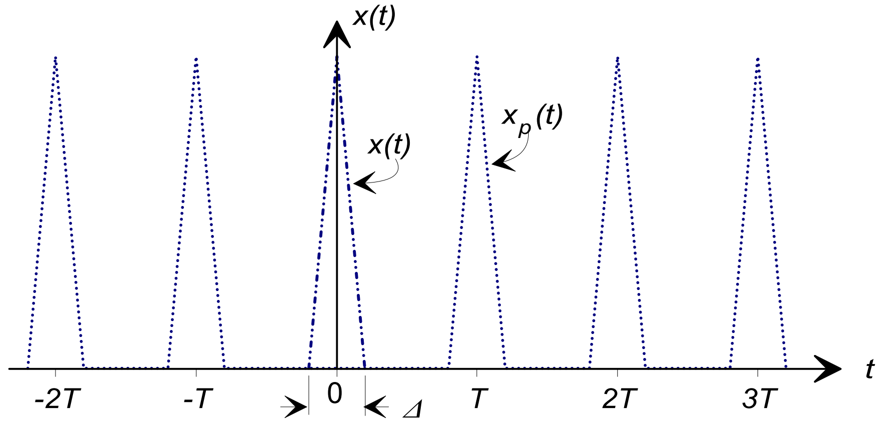
See the class handout for examples.

3 Aperiodic Input Functions - The Fourier Transform

Many waveforms found in practice are not periodic and therefore cannot be analyzed directly using Fourier series methods. A large class of system excitation functions can be characterized as *aperiodic*, or transient, in nature. These functions are limited in time, they occur only once, and decay to zero as time becomes large.

Consider a function $x(t)$ of duration Δ that exists only within a defined interval $t_1 < t \leq t_1 + \Delta$, and is identically zero outside of this interval. We begin by making a simple assumption; namely that in observing the transient phenomenon $x(t)$ within any finite interval that encompasses it, we have observed a fraction of a single period of a periodic function with a very large period; much larger than the observation interval. Although we do not know what the duration of this hypothetical period is, it is assumed that $x(t)$ will repeat itself at some time in the distant future, but in the meantime it is assumed that this periodic function remains identically zero for the rest of its period outside the observation interval.

The analysis thus conjectures a new function $x_p(t)$, known as a *periodic extension* of $x(t)$, that repeats every T seconds ($T > \Delta$), but at our discretion we can let T become very large.



As observers of the function $x_p(t)$ we need not be concerned with its pseudo-periodicity because we will never be given the opportunity to experience it outside the first period, and furthermore we can assume that if $x_p(t)$ is the input to a linear system, T is so large that the system response decays to zero before the arrival of the second period. Therefore we assume that the response of the system to $x(t)$ and $x_p(t)$ is identical within our chosen observation interval. The important difference between the two functions is that $x_p(t)$ is periodic, and therefore has a Fourier series description.

The development of Fourier analysis methods for transient phenomena is based on the limiting behavior of the Fourier series describing $x_p(t)$ as the period T approaches infinity. The derivation proceeds in the following steps (see the class handout for details).

(1) The waveform $x_p(t)$ is described by a Fourier series with lines spaced at intervals

$$\Omega_0 = \frac{2\pi}{T}$$

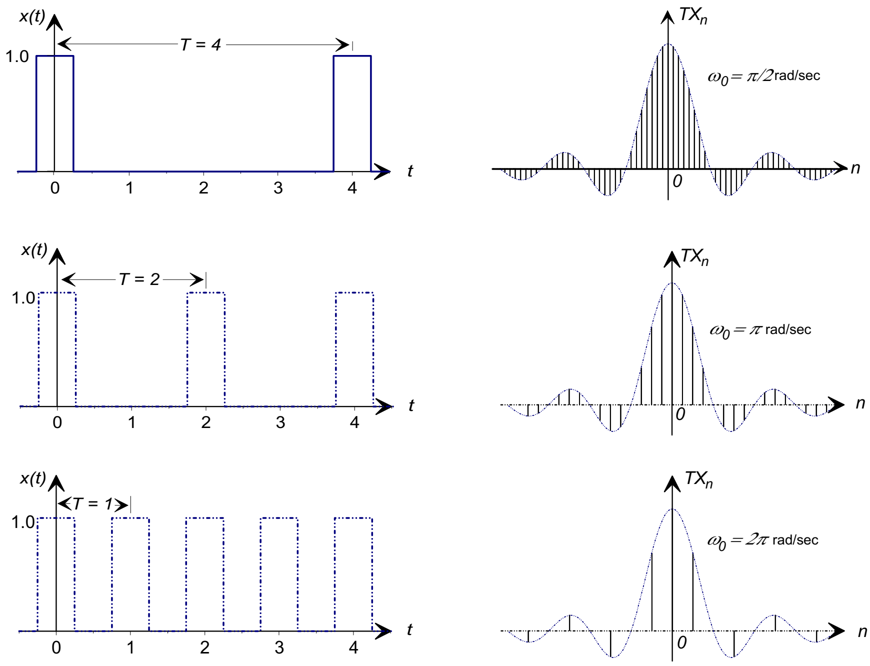
and coefficients

$$X_n = \frac{1}{T}\int_{t_0}^{t_0+T} x_p(t)e^{-jn\Omega_0 t}dt$$

(2) From the synthesis equation (with $t_0 = -T/2$)

$$\begin{aligned} x_p(t) &= \sum_{n=-\infty}^{\infty} X_n e^{jn\Omega_0 t} \\ &= \sum_{n=-\infty}^{\infty} \left\{ \frac{\Omega_0}{2\pi} \int_{-T/2}^{T/2} x_p(t)e^{-jn\Omega_0 t}dt \right\} e^{jn\Omega_0 t} \end{aligned}$$

The figure below shows how the line spectrum varies as the period T changes. Note that the shape of the envelope defining the spectrum is unaltered, but the the magnitude and the line spacing changes.



(c) The period T is now allowed to become arbitrarily large, with the result that the fundamental frequency Ω_0 becomes very small and we write $\Omega_0 = \delta\Omega$. We define $x(t)$ as the limiting case of $x_p(t)$ as T approaches infinity, that is

$$\begin{aligned} x(t) &= \lim_{T\to\infty} x_p(t) \\ &= \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \left\{ \int_{-T/2}^{T/2} x_p(t) e^{-jn\delta\Omega t} dt \right\} e^{jn\delta\Omega t} \delta\Omega \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi} \left\{ \int_{-\infty}^{-\infty} x(t) e^{-j\Omega t} dt \right\} e^{j\Omega t} d\Omega \qquad (4) \end{aligned}$$

where in the limit the summation has been replaced by an integral.

(d) If the function inside the braces is defined to be $X(j\Omega)$, Eq. (**??**) may be expanded into a pair of equations, known as the *Fourier transform pair*:

$$\begin{aligned} X(j\Omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \\ x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega \end{aligned}$$

which are the equations we seek.

(To be continued in Lecture 4)