Plasma Transport theory, 22.616

prob Set #5 Solutions

1. Fluctuation origin of U tensor

$$\underline{\underline{I}}_0 = \int_{-\infty}^{\infty} d^3k\, \delta(\underline{k}\cdot(\underline{v}-\underline{v}'))\frac{\underline{k}\,\underline{k}}{k^4|\epsilon(\underline{k},\underline{k}\cdot\underline{v})|^2}$$

Replace the dielectric constant by a cutoff rule

$$\underline{\underline{I}}_0 = \int_{-\infty}^{\infty} dk_{\parallel} \int_{k_{min}}^{k_{max}} dk_{\perp} k_{\perp} \int_0^{2\pi} d\varphi \; \frac{\delta(k_{\parallel}|\underline{v}-\underline{v}'|)}{(k_{\perp}^2+k_{\parallel}^2)^2}(\underline{k}_{\parallel}+\underline{k}_{\perp})(\underline{k}_{\parallel}+\underline{k}_{\perp})$$

$\parallel$ — parallel to $\underline{v}-\underline{v}'$, $\perp$ — perpendicular to $\underline{v}-\underline{v}'$

First carry out the $k_{\parallel}$ integration

$$\underline{\underline{I}}_0 = \int_{k_{min}}^{k_{max}} dk_{\perp} k_{\perp} \int_0^{2\pi} d\varphi \frac{1}{|\underline{v}-\underline{v}'|}\;\frac{\underline{k}_{\perp}\underline{k}_{\perp}}{k_{\perp}^4}$$

$$\underline{k}_{\perp} = k_{\perp}\cos\varphi\, \underline{e}_x + k_{\perp}\sin\varphi\, \underline{e}_y$$

$$\underline{k}_{\perp}\underline{k}_{\perp} = k_{\perp}^2\cos^2\varphi\, \underline{e}_x\underline{e}_x + k_{\perp}^2\sin^2\varphi\, \underline{e}_y\underline{e}_y + k_{\perp}^2\sin\varphi\cos\varphi\,(\underline{e}_x\underline{e}_y+\underline{e}_y\underline{e}_x)$$

Then carry out the φ integration

$$\underline{\underline{I}}_0 = \pi\int_{k_{min}}^{k_{max}} dk_{\perp}k_{\perp}\,\frac{1}{k_{\perp}^2}\,(\underline{e}_x\underline{e}_x+\underline{e}_y\underline{e}_y)\,\frac{1}{|\underline{v}-\underline{v}'|}$$

$$\underline{e}_x\underline{e}_x + \underline{e}_y\underline{e}_y = \underline{\underline{I}} - \underline{e}_\parallel\underline{e}_\parallel = \underline{\underline{I}} - \frac{(\underline{v}-\underline{v}')(\underline{v}-\underline{v}')}{|\underline{v}-\underline{v}'|^2} = \underline{\underline{U}}\,|\underline{v}-\underline{v}'|$$

So $\underline{\underline{I}}_0 = \cancel{4}\pi \underline{\underline{U}} \int_{k_{min}}^{k_{max}} \frac{dk_\perp}{k_\perp}$

$$= \pi \underline{\underline{U}} \ln \frac{k_{max}}{k_{min}}$$

when $k_{max} = \frac{T}{e^2}$, $k_{min} = \frac{1}{\lambda_{De}}$

$$\Rightarrow \frac{k_{max}}{k_{min}} = \frac{\lambda_{De}}{e^2/T} = \frac{\lambda_{De}\, n_e V_{Te}^2}{n_e e^2 m_e \cancel{V_{Te}}} = 4\pi \frac{\lambda_{De} V_{Te}^2}{\omega_{pe}^2} n_e = 4\pi n_e \lambda_{De}^3$$

$$= \Lambda$$

Therefore $\underline{\underline{I}}_0 = \pi \underline{\underline{U}} \ln \Lambda$

2. Diffusion from plasma waves:

From class discussion

$$\underline{\underline{D}} = \pi \frac{e^2}{m_e^2} \sum_{\underline{k}\omega} \underline{k}\,\underline{k} \langle |\delta\phi_{\underline{k},\omega}|^2 \rangle\, \delta(\omega - \underline{k}\cdot\underline{v})$$

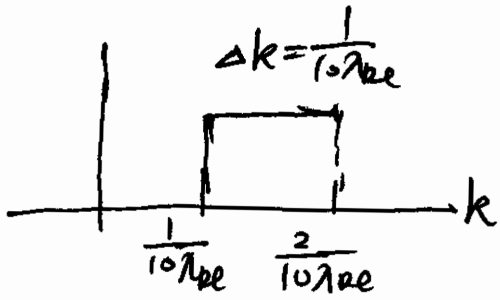
1-D case. $\langle |\delta\phi_{k\omega}|^2 \rangle = \frac{\phi_0^2}{\Delta k}\,\delta(\omega - \omega_{pe})$

$$D = \pi \frac{e^2}{m_e^2} \int dk \int d\omega\, \frac{\phi_0^2 k^2}{\Delta k}\, \delta(\omega - \omega_{pe})\, \delta(\omega - kv)$$

$$= \pi \frac{e^2}{m_e^2} \frac{\cancel{k}\phi_0^2}{\Delta k} \int dk\, k^2 \, \delta(\omega_{pe} - kv)$$

$$= \pi \frac{e^2}{m_e^2} \frac{\phi_0^2}{\Delta k} \frac{\omega_{pe}^2}{v^3}$$

$$= \pi \frac{e^2}{m_e^2} \frac{\phi_0^2 \omega_{pe}^2}{v^3} 10\lambda_{De}$$

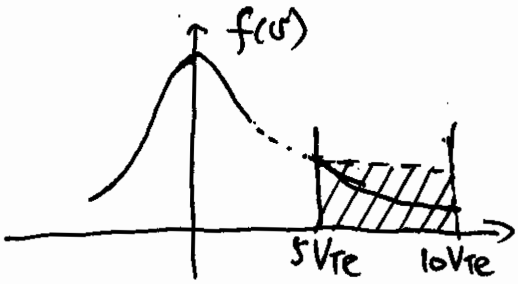


So

Because $v = \frac{\omega_{pe}}{k}$, So v start from $5v_{Te}$ to $10v_{Te}$

~~Let:~~ $\cancel{D = v_{Te}^2 \nu_{ei} = v_{Te}^2}$

Physical effect: flatten the tail of distribution function $f(v)$ ($5v_{Te} \to 10v_{Te}$), heat up the high energy particles, drive a



Current.

On the other hand

$$D = v_{Te}^2 \nu_{ei} = v_{Te}^2 \frac{4\pi n_e e^4}{m_e^2 v^3} \ln\Lambda = \frac{v_{Te}^2}{\cancel{4\pi}} \frac{\omega_{pe}^2}{\cancel{n_e} v^3} \frac{e^2}{m_e} \ln\Lambda$$

i.e.

$$\pi \frac{e^2}{m_e^2} \frac{\phi_0^2 \omega_{pe}^2}{v^3} 10 \lambda_{De} = \frac{v_{Te}^2}{\cancel{4\pi}} \frac{\omega_{pe}^2}{\cancel{n_e} v^3} \frac{e^2}{m_e} \ln\Lambda$$

$$\Rightarrow \quad e^2\phi_0^2 = \frac{1}{10\pi} \frac{m_e v_{Te}^2 e^2}{\lambda_{De}} \ln\Lambda$$

$$= \frac{T_e^2}{10\pi} \frac{1}{\lambda_{De}} \frac{e^2}{T_e} \ln\Lambda$$

$$= \frac{T_e^2}{10\pi} \frac{1}{\Lambda} \ln\Lambda$$

So
$$e\phi_0 = \frac{T_e}{\sqrt{10\pi}} \frac{1}{\sqrt{\Lambda}} \sqrt{\ln\Lambda}$$

we can see, $e\phi_0$ is much smaller than T_e, since Λ is ~~it~~ usually a big number. $\Lambda \sim 10^6 \sim 10^8$, $\log\Lambda \sim 16-18$.

Take the parameter of a typical tokamak.

$T_e = 10\,\mathrm{keV}$, $\Lambda \sim 10^6$, then $\ln\Lambda \sim 16$

$\phi_0 =$ ~~[illegible]~~ (Volt)

4.

3. Correlation time

Estimate the mean square frequency width

$$\langle \omega^2 \rangle = \frac{\sum_{\underline{k},\omega} \omega^2 \langle |\delta\phi_{\underline{k}\omega}|^2 \rangle}{\sum_{\underline{k}\omega} \langle |\delta\phi_{\underline{k}\omega}|^2 \rangle}$$

where the discreteness fluctuations is

$$\langle |\delta\phi_{k\omega}|^2 \rangle = \frac{2e^2}{\pi} \int d^3v' f(\underline{v}') \delta(\omega - \underline{k}\cdot\underline{v}') \frac{1}{k^4 |\epsilon(\underline{k},\omega)|^2}$$

For Simplicity. Assume

$$\epsilon(k,\omega) \simeq 1 + \frac{1}{k^2 \lambda_{De}^2}$$

First Calculate denomenator:

$$\sum_{\underline{k},\omega} \langle |\delta\phi_{\underline{k}\omega}|^2 \rangle$$

$$= \int d^3k\, d\omega \frac{2e^2}{\pi} \int d^3v' f(\underline{v}') \delta(\omega - \underline{k}\cdot\underline{v}') \frac{1}{k^4 |\epsilon(k,\omega)|^2}$$

$$= \frac{2e^2}{\pi} \int d^3k \int d^3v' \frac{f(v')}{k^4 (1 + \frac{1}{k^2 \lambda_{De}^2})^2}$$

$$= \frac{2e^2 n_e}{\pi} \int d^3k \frac{1}{k^4 (1 + \frac{1}{k^2 \lambda_{De}^2})^2}$$

Apply ~~the~~ a cutoff rule. $k_{min} < k < k_{max}$

where $k_{\cancel{min}\,max} = \frac{T}{e^2}$, $k_{min} = \frac{1}{\lambda_{De}}$

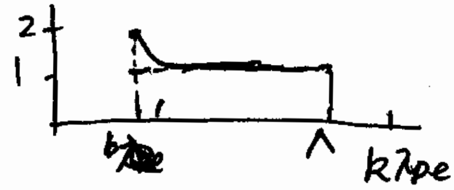
$$\Sigma_{\underline{k},\omega} \langle |\delta\phi_{\underline{k}\omega}|^2 \rangle$$

$$\simeq \frac{2e^2 n_e}{\pi} 4\pi \int_{k_{min}}^{k_{max}} dk \frac{1}{k^2}$$

$$\simeq \frac{2e^2 n_e}{\pi} 4\pi \frac{1}{k_{min}}$$

Note $1 < k\lambda_{De} < k_{max}\lambda_{De} = \Lambda$

So $1 + \frac{1}{k^2\lambda_{De}^2}$ behaves like



~~So~~ Then we calculate

$$\Sigma_{\underline{k},\omega} \langle |\delta\phi_{k\omega}|^2 \rangle \omega^2$$

$$= \int d^3k\, d\omega \frac{2e^2}{\pi} \int d^3v' f(\underline{v}') \frac{\delta(\omega - \underline{k}\cdot\underline{v}')\omega^2}{k^4 |\epsilon(\underline{k},\omega)|^2}$$

$$= \int \cancel{d^3} \frac{2e^2}{\pi} 4\pi \int dk \frac{1}{k^2(1+\frac{1}{k^2\lambda_{De}^2})} \int \underbrace{d^3v' f(v')(\underline{k}\cdot\underline{v}')^2}_{\sim k^2 v_{Te}^2 n_e}$$

$$\simeq \frac{2e^2}{\pi} 4\pi v_{Te}^2 n_e \int_{k_{min}}^{k_{max}} dk$$

$$\simeq \frac{2e^2}{\pi} 4\pi v_{Te}^2 k_{max} n_e$$

Therefore $\langle \omega^2 \rangle = v_{Te}^2 k_{max} k_{min} = v_{Te}^2 k_{min}^2 \frac{k_{max}}{k_{min}}$

$$= \frac{v_{Te}^2}{\lambda_{De}^2} \Lambda = \omega_{pe}^2 \Lambda$$

So $\tau_c \sim \frac{1}{\sqrt{\Lambda}}\frac{1}{\omega_{pe}}$, so the correlation time is very short.

If we only consider $\lambda \sim \lambda_{max} = \lambda_{De}$. $\tau_c = \frac{\lambda_{De}}{v_{Te}} = \frac{1}{\omega_{pe}}$

If $\lambda \sim \lambda_{min} = \frac{\lambda_{De}}{\Lambda}$, $\tau_c \sim \frac{1}{\omega_{pe}\sqrt{\Lambda}}$.

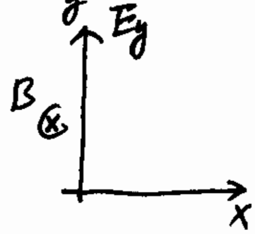
If the wavelength spectrum is flaat for from $\frac{\lambda_{De}}{\Lambda}$ to λ_{De}.

we obtain the geometric mean of these two limits

The collisional effects become important within a debye length

These collisions significadntly short the correllation time $\left(\frac{1}{\sqrt{\Lambda}}\right)$

4. Solution:



For a slab geometry plasma,

when $\omega \ll \Omega_i$, we consider guiding center fluctuations

$$\delta\phi(x,t) = \sum_{k_y k_z \omega} \delta\phi_{\underline{k}\omega}\, e^{ik_y y + ik_z z - i\omega t}$$

$E \times B$ drift

$$\delta v_x(t) = \frac{c}{B}\delta E_y = \frac{c}{B}\sum_{k_y,k_z,\omega} -ik_y\,\delta\phi_{\underline{k}\omega}\exp\left(ik_y y(t) + ik_z z(t) - i\omega t\right)$$

~~$\delta x\delta x =$~~ Then

$$\delta x \delta x = \int_0^t dt' \delta v_x(t') \int_0^t dt'' \delta v_x(t'')$$

$$= \frac{c^2}{B^2} \int_0^t dt' \int_0^t dt'' \sum_{k_y k_z \omega} \sum_{k_y' k_z' \omega'} (-k_y^2)\, \delta\phi_{\underline{k}\omega}\, \delta\phi_{\underline{k}'\omega'}$$

$$e^{ik_y y(t') + ik_z z(t') - i\omega t'} \times e^{ik_y' y(t'') + ik_z' z(t'') - i\omega' t''}$$

But $\langle \delta\phi_{\underline{k}\omega}\, \delta\phi_{\underline{k}'\omega'} \rangle = \langle |\delta\phi_{\underline{k}\omega}|^2 \rangle\, \delta(\underline{k}+\underline{k}')\,\delta(\omega+\omega')$

Therefore

$$\langle \delta x \delta x \rangle = \frac{c^2}{B^2} \int_0^t dt' \int_0^t dt'' \sum_{k_y k_z \omega} k_y^2 \langle |\delta\phi_{\underline{k}\omega}|^2 \rangle\, e^{ik_y(y(t') - y(t''))}$$

$$\times e^{ik_z(z(t') - z(t''))} \times e^{-i\omega(t'-t'')}$$

For unperturbed ~~coordinate~~ orbit:

$$\begin{cases} y(t) = y_0 \\ z(t) = z_0 + V_z t \end{cases}$$

So $$\langle \delta x \delta x \rangle = \frac{c^2}{B^2} \int_0^t dt' \int_{-t'}^{t-t'} d\tau \sum_{k_y k_z \omega} k_y^2 \langle |\delta\phi_{\underline{k}\omega}|^2 \rangle\, e^{-ik_z V_z \tau + i\omega\tau}$$

$(\tau = t'' - t')$

$$= 2\pi t \sum_{k_y k_z \omega} k_y^2 \langle |\delta\phi_{\underline{k}\omega}|^2 \rangle\, \delta(\omega - k_z V_z)\, \frac{c^2}{B^2}$$

According definition

$$D_{xx} = \frac{1}{2}\left\langle \frac{\delta x\,\delta x}{\Delta t} \right\rangle$$

$$= \pi \sum_{k_y k_z \omega} k_y^2 \langle |\delta\phi_{k\omega}|^2 \rangle \,\delta(\omega - k_z v_z)\frac{c^2}{B^2}$$

Assume adiabatic response of electrons

$$\delta n_e = n\frac{e}{T}\delta\phi$$

$$\Rightarrow \quad \delta\phi = \frac{T}{en}\,\delta n_e \quad \Rightarrow \delta\phi_{k\omega} = \frac{T}{en}\,\delta n_{k\omega}$$

Then

$$D_{xx} = \pi\left(\frac{cT}{eB}\right)^2 \sum_{k_y k_z \omega} k_y^2 \left\langle \left|\frac{\delta n_{k\omega}}{n}\right|^2 \right\rangle \delta(\omega - k_z v_z)$$

Now make a quantitative estimate of

$$\left\langle \left|\frac{\delta n}{n}\right|^2 \right\rangle \equiv \sum_{k_y k_z \omega} \left\langle \left|\frac{\delta n_{k\omega}}{n}\right|^2 \right\rangle$$

Assume the drift wave spectrum has a frequency width

$$\Delta\omega \simeq \omega_{*e} = k_y \rho_i v_{Ti}/L_n$$

Then $D_{xx} = \pi\left(\frac{cT}{eB}\right)^2 \frac{k_y^2}{\omega_{*e}} \left\langle \left|\frac{\delta n}{n}\right|^2 \right\rangle$

$$\Rightarrow \quad \left\langle \left|\frac{\delta n}{n}\right|^2 \right\rangle = D_{xx} \frac{1}{\pi} \left(\frac{eB}{cT}\right)^2 \frac{\omega_{*e}}{k_y^2}$$

$$= D_{xx} \frac{1}{\pi} \left(\frac{\omega_{ci}}{V_{Ti}^2}\right)^2 \frac{1}{k_y^2} k_y \rho_i \frac{V_{Ti}}{L_n}.$$

$$= \frac{D_{xx}}{\pi} \frac{1}{V_{Ti}^2} \left(\frac{\omega_{ci} \rho_i}{V_{Ti}}\right)^2 \frac{1}{k_y \rho_i} \frac{V_{Ti}}{L_n}$$

$$\boxed{\begin{array}{l} k_y \rho_i \sim 1 \\ \omega_{ci} \rho_i \sim V_{Ti} \end{array}}$$

$$\simeq \frac{D_{xx}}{\pi} \frac{V_{Ti}}{L_n} \frac{1}{V_{Ti}^2}$$

$$\simeq \frac{1}{\pi} \times 10^5 \times \frac{1}{(2 \times 10^5)^2}$$

$$\simeq \cancel{8 \times 10^{-5}} \; \cancel{8}\, 7.96 \times 10^{-7}$$

$$\Rightarrow \quad \frac{\delta n}{n} \sim \sqrt{\left\langle \left|\frac{\delta n}{n}\right|^2 \right\rangle} \sim 0.09\,\%$$