PROFESSOR: All right. Hi, everyone.

AUDIENCE: Hi.

PROFESSOR: We're getting towards the end of the semester. Things are starting to cohere and come together. We have one more midterm exam. So there is an exam next Thursday, the 18th. OK? There will be a problem set due. It'll be posted later today, and it will be due next week on Tuesday as usual. Of course, next week on Tuesday is a holiday technically, so we'll actually make the due date be on Wednesday. So on Wednesday at 10 o'clock.

You should think of this problem set as part of the review for the exam. Material that is covered today and on Thursday will be fair game for the exam.

So the format of the exam is going to be much more canonical. It's going to be a series of short answer plus a series of computations. They'll be, roughly speaking, at the level of the problem sets, and a practice exam will be posted in the next couple of days. The practice exam is going to be of the same general intellectual difficulty, but it's going to be considerably longer than the actual exam.

So as a check for yourself here's what I would recommend. I would recommend sitting down and giving yourself an hour and a half or two hours with the practice exam and see how that goes. OK? Try to give yourself the time constraint. And here's one of the things that-- there's a very strong correlation between having nailed the problem sets, and worked through all the practice exams, and just done a lot of problems, and doing well on exams. You'll be more confident. The only way to study for these things is just do a lot of problems.

So on Stellar, for example, are a bunch of previous problem sets, and previous exams and practice exams from previous years. I would encourage you to look at those also on OCW and use those as practice. OK, any practical questions before we get on to the physics. Yeah?

AUDIENCE: Will the exam cover all the material, or all the material [INAUDIBLE]?

PROFESSOR: This is a cumulative topic. So the question is does it cover everything, or just last couple of weeks. And it's a cumulative topic, so the entire semester is necessary in order to answer the problems. Other questions? OK. Anything else?

So quick couple of review questions before we launch into the main material of today. These are going to turn out to be useful reminders later on in today's lecture. So first, suppose I tell you I have a system with energy operator E, which has an operator A such that the commutator of E with A would say plus h bar A. OK? What does that tell you about the spectrum of the energy operator?

AUDIENCE: It's a ladder.

PROFESSOR: It's a ladder, exactly. So this tells you that the spectrum of E, or the energy eigenvalues En, are evenly spaced-- and we need a dimensional constant here, omega-- evenly spaced by h bar omega. And more precisely, that given a state phi E, we can act on it with the operator A to give us a new state, which is also an energy eigenstate, with energy E plus h bar omega. right? So any time you see that commutation relation, you know this fact to be true.

Second statement. Suppose I have an operator B which commutes with the energy operator. OK? So that the commutator vanishes. What does that tell you about the system?

AUDIENCE: Simultaneous eigenfunctions [INAUDIBLE].

PROFESSOR: Excellent. So one one consequence is that there exists simultaneous eigenfunctions phi sub E, B, which are simultaneous eigenfunctions of both E and B.

What else does it tell you? Well, notice the following. Notice that if we took this

computation relation and set omega to 0, we get this commutation relation. So this commutation relation is of the form of this commutation relation with omega equals 0. So what does that tell you about the states? About the energy eigenvalues? What happens if I take a state phi sub E and I act on it with B? What do I get? What can you say about this function?

Well, what is its eigenvalue under E? It's E. It's the same thing, because they commute. You can pull the E through, multiply it, you get the constant E, you pull it back out. This is still an eigenfunction of phi of the energy operator with the same eigenvalue. But is it necessarily the same eigenfunction? No, because B may act on phi and just give you a different state. So this is some eigenfunction with the same energy, but it may be a different eigenfunction. OK?

So let's think of an example of this. An example of this is consider a free particle. In the case of a free particle, we have E is equal to P squared upon 2m. And as a consequence E and P commute. Everyone agree with that? Everyone happy with that statement? They commute. So plus 0 if you will.

And here what I wanted to say is when you have an operator that commutes with the energy, then there can be multiple states with the same energy which are different states, right? Different states entirely. So for example, in this case, what are the energy eigenfunctions? Well, e to the ikx. And this has energy h bar squared k squared upon 2m.

But there's another state, which is a different state, which has the same energy. e to the minus ikx. OK? So when you have an operator that commutes with the energy operator, you can have simultaneous eigenfunctions. And you can also have multiple eigenfunctions that have the same energy eigenvalue, but are different functions. For example, this. Everyone cool with that?

Now, just to make clear that it's actually the commuting that matters, imagine we took not the free particle, but the harmonic oscillator. E is p squared upon 2m plus m omega squared upon 2 x squared. Is it true that E-- I'll call this harmonic oscillator-- does E harmonic oscillator commute with P? No, you're going to get a

term that's m omega squared x from the commutator. The potential is not translationally invariant. It does not commute with momentum. So this is not equal to 0.

So that suggests that this degeneracy, two states having the same energy, should not be present. And indeed, are the states degenerate for the harmonic oscillator. No. No degeneracy. Yeah?

AUDIENCE: We did something [INAUDIBLE] where I think it said that the eigenfunctions were complete.

PROFESSOR: Yes.

AUDIENCE: What does that mean?

PROFESSOR: What does it mean for the eigenfunctions to be complete? What that means is that they form a basis.

AUDIENCE: So the basis doesn't necessarily mean not [INAUDIBLE].

PROFESSOR: Yeah, no one told you the basis had to be degenerate, and in particular, that's a excellent-- so the question here is, wait a minute, I thought a basis had to be a complete set-- if you had an energy operator and you constructed the energy eigen-- this is a very good question. Thank you. If I have the energy operator and I construct it's energy eigenfunctions, then those energy eigenfunctions form a complete basis for any arbitrary function. Any function can be expanded in it, right?

So for example, for the free particle, the energy eigenfunctions are e to the ikx, the momentum eigenfunctions for any value of k. But wait, how can there be multiple states with the same energy? Isn't that double counting or something? And the important thing is those guys have the same energy, but they have different momenta. They're different states. One has momentum h bar k. One has momentum minus h bar k. And you know that that has to be true, because the Fourier theorem tells you, in order to form a complete basis, you need all of them, all possible values of k. So there's no problem with being a complete basis and

having states have the same energy eigenvalue, OK? It's a good question. Other questions? Yeah?

AUDIENCE: So if we have a potential that only admits bound states, we'll never have this commutation happen basically?

PROFESSOR: Yeah, exactly. Excellent. So the observation is this. Look, imagine I have a potential that's not trivial. It's not 0, OK? Will the momentum commute with the energy operator. No, because it's got a potential that's going to be acted upon by P, so you'll get a derivative term.

But more precisely, if I have a system with bound states, I have to have a potential, right? And then I can't have P commuting with the energy operator, which means I can't have degeneracies. So indeed, if you have bound states, you cannot have degeneracies. That's exactly right. Yeah?

AUDIENCE: But doesn't this break down in higher dimensions?

PROFESSOR: Excellent, so we're going to come back to higher dimensions later. So the question predicts what's going to happen in the rest of the lecture. What we're going to do in just a minute is we're going to start working in three dimensions for the first time. We're going to leave 1D behind. We're going to take our tripped out tricycle and replace it with a Yamaha. As you'll see, it has the same basic physics driving its, well, self. It's the same dynamics.

But I want to emphasize a couple things that are going to show up. So the question is, isn't this story different in three dimensions? And we shall see exactly what happens in higher dimensions. We'll work in two dimensions. We'll work in three dimensions. We'll work in more. Doesn't really matter how many dimensions we work in. You'll see it.

OK, third thing. You studied this in some detail in your problem set. Suppose I have an energy operator that commutes with a unitary operator, U, OK? So it commutes to 0. And U is unitary, so U dagger U is one, is the identity.

So what does this tell you? Well, first off from these guys it tells us that we can have simultaneous eigenfunctions. It also tells us too that if we take our state phi and we act on it with U, this could give us a new state, phi tilde sub E, which will necessarily have the same energy eigenvalue, because U and E commute. But it may be a different state. We'll come to this in more detail later.

But the third thing, and I want to emphasize this, is this tells us, look, we have a unitary operator. We can always write the unitary operator as e to the i of a Hermitian operator. So what is the meaning of the Hermitian operator? What is this guy?

So in your problem set, you looked at what unitary operators are. And in the problem set, it's discussed in some detail that there's a relationship between a unitary transformation, or unitary operator, and the symmetry. A symmetry is when you take your system and you do something to it, like a rotation or translation, and it's a symmetry if it doesn't change anything, if the energy remains invariant. So if the energy doesn't change under this transformation, we call that a symmetry.

And we also showed that symmetries, or translations, are generated by unitary operators. For example, my favorite examples are the translate by L operator, which is unitary. And also e to the minus dxl. And the boost by q operator, which similarly is e to the minus qdp. And the time translation operator, U sub t, which is equal to e to the minus i t over h bar energy operator. OK?

So these are transformation operators. These are symmetry operators, which translate you by L, boost or speed you up by momentum q, evolve you forward in time t. And they can all be expressed as e to the unitary operator. So this in particular is I i over h bar p. And this is similarly x.

And earlier we understood the role of momentum having to do translations in the following way. There's a beautiful theorem about this. If you take a system and its translation invariant, the classical statement of Noether's theorem is that there's a conserved quantity associated with that translation. That conserved quantity is the momentum. And quantum mechanically, the generator of that transformation the

Hermitian operator that goes upstairs in the unitary is the operator associated to that conserved quantity, associated to that observable.

You have translations. There's a conserved quantity, which is momentum. And the thing that generates translations, the operator that generates translations, is the operator representing momentum.

So each of these are going to come up later in today, and I just wanted to flag them down before the moment. OK, questions before we move on? Yeah?

AUDIENCE: So you made the claim that every unitary operator can be expressed as p to the eigenfunction.

PROFESSOR: OK, I should be a little bit careful, but yes. That's right.

AUDIENCE: But if I take the [INAUDIBLE] I should be able to figure out what it is, but you can't take the [INAUDIBLE]

PROFESSOR: The more precise statement is that any unitary-- any one parameter family of unitary operators can be expressed in that form. And then you can take a derivative. And that's the theory of [INAUDIBLE], which is beyond the scope. Let me make a very specific statement, which is that one parameter of [INAUDIBLE] unitary transformation. So translations by l, where you can vary l, can be expressed in that form. And that's a very general statement.

OK, so with all that as prelude, let's go back to 3D. So in 3D, the energy operator-- so what's going to change? Now instead of just having position and its momentum, we now also have-- I'll call this P sub x-- we can also have a y-coordinate and we have a z-coordinate. And each of them has its momentum. P sub z and P sub y.

And here's just a quick practical question. We know that x with Px is equal to i h bar. So what do you expect to be true of x with y?

AUDIENCE: 0.

PROFESSOR: Why?

AUDIENCE: [INAUDIBLE].

PROFESSOR: What does this equation tell you? What is its physical content? Well, that they don't commute, good. What does that tell you physically? Yes?

AUDIENCE: That there's an uncertainty principle connecting the two.

PROFESSOR: Excellent. So that's one statement. So the consequence of this is that there's an uncertainty principle. Delta x delta Px must be greater than or equal to h bar upon 2. What's another way of saying this? Do there exist simultaneous eigenfunctions of x and P?

AUDIENCE: No.

PROFESSOR: No. No simultaneous eigenfunctions. OK, so you can't have a definite value of x and a definite value of P simultaneously. There's no such state. It's not that you can't know. It's that there's no such state.

Do you expect to be able to know the position in x and the position in y simultaneously? Sure. OK, so this turns out to be 0. And in some sense, you can take that as a definition of quantum mechanics. x and y need to be 0. And similarly, Px and Py commute. The momenta are independent. However, Py and y should be equal to minus i h bar. Good. Exactly.

So the commutators work out exactly as you'd naively expect. Every pair of position and its momenta commute canonically to i h bar. And every pair of coordinates commute to 0. Every pair of momenta commute to 0. Cool?

So what kind of systems are we going to interested in? Well, we're going to be interested in systems where the energy operator is equal to P vector hat squared upon 2m plus U of x and y and z, hat, hat. You can see why dropping the hats becomes almost /

So in this language, we can write the Schrodinger equation. This is just a direct extension of the 1D Schrodinger equation. i h bar dt of psi. Now our wave function is

a function of x and y and z. There's some finite probability then to find a particle at some position. That position is labeled by the three coordinates. Is equal to-- and of t. Is equal to-- well, I'm actually write this in slightly different form.

This is going to be easier if I use vector notation. So I'm going to write this as psi of r and t, where r denotes the position vector, is equal to the energy operator acting on it. And P is just equal to minus i h bar the gradient. So this is minus i h bar squared, or minus h bar squared upon 2m gradient squared plus u of x or now u of r psi of r and t.

Quick question, what are the units or what are the dimensions of psi of r in 3D?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yeah, one over length to the root three halves. And the reason is this norm squared gives us a probability density, something that when we integrated over all positions in a region integral d 3x is going to give us a number, a probability. So its actual magnitude must be-- or its dimension must be 1 over L to the 3/2. Just the cube of what it was in 1D.

And as you'll see on the problems set and as we'll do in a couple lectures down the road, it's convenient sometimes to work in Cartesian, but it's also sometimes convenient to work in spherical coordinates. And it does not matter. And here's a really deep statement that goes way beyond quantum mechanics. It does not matter which coordinates you work in. You cannot possibly get a different answer by using different coordinates. So we're going to be ruthless in exploiting coordinates that will simplify our problem throughout the rest of this course.

In the notes is it a short discussion of the form of the Laplacian, or the gradient squared in Cartesian spherical and cylindrical coordinates. You should feel free to use any coordinate system you want at any point. You just have to be consistent about it.

So let's work out a couple of examples. And here are all we're going to do is apply exactly the same logic that we see over and over in 1D to our 3D problems. So the

first example is a free particle in 3D. So before I get started on this, any questions? Just in general 3D questions? OK.

So this stuff starts off easy. And I'm going to work in Cartesian coordinates. And a fun problem is to repeat this analysis in spherical coordinates, and we'll do that later on. OK, so free particle in 3D, so what is the energy eigenfunction equation look like? We want to find-- the Schrodinger equation has exactly the same structure as before. It's a linear differential equation. So if we find the eigenfunctions of the energy operator, we can use superposition to construct the general solution, right?

So exactly as in 1D, I'm going to construct first the energy eigenfunctions , and then use them in superposition to find a general solution to the Schrodinger equation. OK? So let's construct the energy eigenfunctions.

So what is the energy eigenvalue equation look like? Well, E on psi is equal to minus h bar squared upon 2m. And in Cartesian, the Laplacian is derivative respect to x squared plus derivative with respect to y squared plus derivative with respect to z squared. And we have no potential, so this is just psi. So that would be energy operator acting on it. And the eigenvalue equation is at a constant, the energy E on psi satisfies this equation.

I'm going to write this phi sub e to continue with our notation of phi being the energy eigenfunctions. It of course, doesn't matter what I call it, but just for consistency I'm going to use the letter phi.

So this is a very easy equation to solve. In particular, it has a lovely property, which is that it's separable. OK? So separable means the following. It means, look, I note, I just observed that this differential equation can be written as a sum of terms where there's a derivative with respect to only one variable. There's a differential operator with respect to only one variable and another differential operator with respect to only one variable added together. And when you see that, you can separate.

And here's what I mean by separate. I'm going to just construct. I'm not going to say that this is a general solution. I'm just going to try to construct a set of solutions of

the following form. Psi E of x, y, and z is equal to psi E-- I will call this psi sub x of x times phi sub y of y times phi sub z of z.

And if we take this and we plug it in, let's see what we get. This gives us that E on phi E is equal to minus h bar squared upon 2m. Well the dx squared acting on phi sub e is only going to hit this guy. So I'm going to get phi x prime prime phi y phi z. Plus from the next term phi y prime prime phi x phi z. And from the next term phi z prime prime phi x phi y.

But now I can do a sneaky thing and divide the entire equation by phi sub e. Phi sub e is phi x phi y phi z. So if I do so, I lose the phi y phi z, and I divide by phi x. And in this term, when I divide by phi x phi y phi z, I lose the x and z, and I have a left over phi sub y, or phi y. And similarly here, phi sub z upon phi z. Everyone cool with that? Yeah?

AUDIENCE: Can we also lose the [INAUDIBLE]?

PROFESSOR: I don't think so. Minus h bar squared over 2m just hangs out for the ride. So when I take the derivatives, I get these guys. And I have E times the function. We could certainly write this as 2m over h bar squared and put it over here. That's fine.

OK, so this is the form of the equation we have, and what does this give us? What content does this give us? Well, note the following. This is a funny system. This is a function of x. This is a function of y, g of y only, and not of x or z. And this is a function only of z, and not of x or y. Yeah?

So we have that E, and let's put this as minus 2m over h bar squared is equal to a function of x plus a function of y plus a function of h. What does this tell you?

AUDIENCE: They're all constant.

PROFESSOR: They're all constant, right. So the important thing is this equation has to be true for every value of x, y, and z. It's a differential equation. It's true everywhere. It's true here. It's true there. It's true at every point. Yeah?

So for any value of x, y, and z, this equation must be true. So now imagine I have a

particular solution g at h. I'm going to fix y and z to some particular point. I'm going to look right here. And here that fixes y and z. So these are just some numbers. And suppose we satisfy this equation. Then there's a very x leaving y and z fixed. What must be true of f of x?

AUDIENCE: Constant.

PROFESSOR: It's got to be constant, exactly. So this tells us that f of x is a constant. I.e. phi x prime prime of over phi x is a constant. I'll call it epsilon x. And phi y prime prime of over phi y and this is a function of y of x is equal to epsilon y. And actually for-- yeah, that's fine. For fun I'm going to put in a minus. It doesn't matter what I call this coefficient. And similarly for phi z prime prime z over phi z is equal to minus epsilon z.

So this tells us that minus 2m upon h bar squared e is equal to minus epsilon x minus epsilon y minus epsilon z. And any solutions of these equations with some constant value of epsilon x, epsilon y, and epsilon z is going to give me a solution of my original energy eigenvalue equation, where the value of capital E is equal to the sum. And I can take the minus signs make this plus plus. Yeah?

AUDIENCE: Can epsilon x, epsilon y, and epsilon z-- can one of them be negative if the other's are sufficiently positive or vice versa? Or is that [INAUDIBLE]?

PROFESSOR: Let's check. Let's check. So what are the solutions of this equation? Yeah. So solutions to this equation phi double-- so let's write this in a slightly more familiar form. This says that phi prime prime plus epsilon x phi is equal to 0, OK? But this just tells you that phi is exponential. Phi is equal to a e to the Ikx kxx plus B e to the minus ikxx, where k squared is equal to kx squared is equal to epsilon x. OK? So this becomes-- and similarly for epsilon y and epsilon z, each with their own value of ky and kz who squares the epsilon accordingly.

So what can you say about these epsilons? Well, the epsilons are strictly positive numbers. So to answer your question. So the epsilons have to be positive.

OK, so this equation becomes, though, E is equal to-- I'm going to put the h bar squared over 2m back on the other side. h bar squared upon 2m. Epsilon x, epsilon y, epsilon z. But those are just Kx squared plus Ky squared plus Kz squared. Kx squared plus Ky squared plus Kz squared, also known as h bar squared upon 2m k vector squared, where the wave function, phi sub E is equal to some overall normalization constant times, for the first function-- where did the definition go? Right here. So from here, phi x is the exponential with Kx. This is an exponential y with Ky. And then the exponential in z with Kz. e to the ikx times x plus Ky times y plus Kz times z. Let me write this as e to the i. e to the i. Also known as some normalization constant e to the i k vector dot r vector. OK?

So the energy, if we have a free particle, the energy eigenfunctions can be put in the form, or at least we can build energy eigenfunctions of the form, plane waves with some 3D momentum with energy E is equal to h bar squared k squared upon 2m, just as we saw for the free particle in one dimension. And the actual wave function is nothing but a product of wave functions in 1D. Yeah?

AUDIENCE: What happened to the minus ikx minus iky minus ikz terms?

PROFESSOR: Good, so here I just dropped these guys. So I just picked examples where we just picked e to the ikz. That's an excellent question. So I've done something here. In particular, I looked at a special case. And here's an important lesson from the theory of the separable equations, which is that once I separate-- so if I have a separable equation and I find it separated solution, phi E is equal to phi x of x, phi y of y phi z of z, not all functions-- not all solutions of the equation are of this form. They're not. I had to make that assumption. I said suppose it's of this form right here. So this is an assumption. OK?

However, these form a good basis. By taking suitable linear combinations of them, suitable super positions, I can build a completely general solution. For example, as was noted, the true solution of this equation, even just focusing on the x, is a superposition of plus and minus waves, waves with plus positive negative momentum.

So how do we get that? Well, we could write that as phi is equal to e to the ikx phi y phi z plus e to the minus ikx phi y phi z with the same phi y and phi z. So these are actually in there. Yeah?

AUDIENCE: My other question is so I still don't see why any of the epsilons can't have a negative sign. You have an exponential, a real exponential as one of your products.

PROFESSOR: OK, so if we had a negative epsilon, is that wave function going to be normalizable?

AUDIENCE: Oh, as r goes to-- but can you just keep the minus term?

PROFESSOR: In which direction?

AUDIENCE: Oh, right.

PROFESSOR: So if it's converging in this direction, it's got to be growing in this direction. And that's not going to be normalizable. And so as usual with the plane wave, we can pick the oscillating solutions that are also not normalizable to one, but they're delta function normalizable. And so that's what we've done here. It's exactly the same thing as in 1D. Yeah?

AUDIENCE: So does this mean that any superposition of plane waves with wave vector equal in magnitude will also be an eigenfunction of the same energy?

PROFESSOR: Absolutely. Awesome. Great observation. So the observation is this. Suppose I take K, I'll call it K1. So this is a vector such that K1 squared times h bar squared over 2m is equal to E.

Now there are many vectors k which have the same magnitude, but not the same direction. So we could also make this equal to h bar squared K2 squared vector squared over 2m where K2 is not equal to K1 as a vector, although they share the same magnitude.

So that's interesting. So that looks a lot like before. In 1D, we saw that if we have k or minus k, these have the same energy. All right? Now if we have any K, K1-- so this is 1D. In 3D, if we have K1 and K2 with the same magnitude and the same

energy, they're degenerate. That's interesting.

Why? Why do we have this gigantic degeneracy of the energy eigenfunctions for the free particle in three dimensions? Yeah?

AUDIENCE: Well, there are an infinite number of directions it could be going in with the same momentum.

PROFESSOR: Awesome. So this is clearly true that there are an infinite number of momenta with the same magnitude. So there are many, many, but why? Why do they have the same energy? Couldn't they have different energy? Couldn't this one have a different energy?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Excellent, it's symmetric. The system is invariant under rotation. Who are you to say this is the x direction? I call it y. Right? So the system has a symmetry. The symmetry is rotations. And when we have a symmetry, that means there's an operator, a unitary operator, which affects that rotation, the rotation operator. It's the operator that takes a vector and rotates in some particular way.

We have a unitary operator that's a symmetry that means it commutes with the energy operator. But if it commutes with the energy operator, we get can degeneracies. We can get states that are different states mapped to each other under our unitary operator, under our rotation. We get states which are different states manifestly. But which have the same energy, which are shared energy eigenvalues. Cool?

And this is a really lovely example, both in 1D and 3D, that when you have a symmetry, you get degeneracies. And when you have a degeneracy, you should be very suspicious that there's a symmetry hanging around, lurking around ensuring it, OK? And this is an important general lesson that goes way beyond the specifics of the free particle. Yeah?

AUDIENCE: So that occurs in systems with bound states [INAUDIBLE]?

PROFESSOR: Yeah, it occurs in systems with bound states and systems with non bound states. So here we're talking about a free particle. Certainly not bound. And its true. For bound states, we'll also see that there will be a degeneracy associated with symmetry.

Now your question is a really, really good one, because what we found-- let me rephrase the question. The question is, look, in 1D when we had bound states, there was no degeneracy. Didn't matter what you did to the system. When you had bound states, bound states were non degenerate.

In 3D, we see that when you have a free particle, you again get degeneracy. In fact, you get a heck of a lot more degeneracy. You get a sphere's worth. Although actually, that's a sphere in 0 dimension, right? It's a 0 dimensional sphere, two points. So you get a sphere's worth of degenerate states for the free particle. Well, what about bound states? Are bound states non degenerate still? Fantastic question. Let's find out.

So let's do the harmonic oscillator. Let's do the 3D harmonic oscillator to check. So the 3D harmonic oscillator, the potential is h bar squared. And let's pick for fun the rotationally symmetric 3D harmonic oscillator. m omega squared upon 2 x squared plus y squared plus z squared. This could also be written m omega squared upon 2 r squared.

So I could write it in spherical coordinates or in Cartesian coordinates. This is really in vector notation. Doesn't matter. It's the same thing.

3D harmonic oscillator. So what you immediately deduce about the form of the energy eigenfunctions? Well, we have that E phi E is equal to P squared over minus h bar squared over 2m-- so here's the energy eigenvalue equation-- times dx squared plus dy squared plus dz squared plus m omega squared upon 2 times x squared plus y squared plus z squared phi E.

But I can put this together in a nice way. This is minus h bar squared upon 2m dx squared plus m omega squared upon 2 x squared plus ditto for y plus ditto for z phi E. Yeah? Everyone agree?

So this is differential operator that only involves x. Doesn't involve y or z. Ditto y, but no x or z. And ditto z, but no x or y. Aha, this is separable just as before. So now we have a nice separable system where I want to solve the equations 3 times, once for x, y, and z. And I'm just going to write it for x, y, and z epsilon sub x phi x is equal to minus h bar squared over 2m dx squared plus m omega squared upon 2 x squared phi sub x. And ditto for x, y, z. And then phi E is going to be equal to phi x of x, phi y of y, and phi z of z, OK? Where E is equal to epsilon x plus epsilon y plus epsilon z. Note that I've used a slightly different definition of epsilon here as before. Here it's explicitly the energy eigenvalue.

So what this is telling is, look, we know what this equation is. This equation is the same equation we ran into in the 1D harmonic oscillator. It's exactly the 1D harmonic oscillator problem. So the solution to the 3D harmonic oscillator problem can be written for energy eigenfunctions, can be constructed by taking a harmonic oscillator in the x direction-- and we know what those are. There's a tower of them. There's a ladder of them created by the raising operator and lowered by the lowering operator. And similarly for y, and similarly for z.

So I'm going to write this slightly differently in the same place as phi sub E is equal to phi sub nx of x. The state with energy E sub nx. Phi sub ny of y. Phi sub nz of z. Where these are all the single dimension, one dimensional harmonic oscillator eigenfunctions. And E is equal to Ex E sub nx plus E sub ny plus E sub nz. OK?

So let's look at the consequences of this. So first off, does anyone have any questions? I went through the kind of quick. Any questions about that? If you're not comfortable with separable equations, you need to become super comfortable with separable equations. It's an important technique. We're going to use it a lot.

So the upshot is that if I write-- in fact, I'm going to write that in slightly different notation. If I write phi E is equal to phi n of x, where it's the-- and phi l of y and phi m of z-- actually that's a stupid ordering. Let's try that again. l, m, n. That is the alphabetical ordering. With the energy is now equal to-- we know the energy is of a state with of the harmonic oscillator with excitation number l. It's h bar omega, the

overall omega, times l plus 1/2.

But from this guy it's got excitation number m, so energy of that is h bar omega m plus 1/2 plus m. And so now that's plus 1. And for this guy similarly, h bar omega n plus n and now plus 1/2 again plus 3/2. This is a basis of solutions of the energy eigenfunction equations. These are the solutions of the energy eigenfunctions for the 3D harmonic oscillator.

And now here's the question. The question that was asked is, look, there are no degeneracies in bound states in 1D. Here we have manifestly a 3D bound state system. Are there degeneracies?

AUDIENCE: Yes.

PROFESSOR: Yeah, obviously, right? So for example, if I call l1 this 0 and this 0. Or if I call this 010 or 001, those all have the same energy. They have the energy h bar omega 0 times 1 plus 3/2 or 5/2.

So let's look at this in a little more detail. Let's write a list of the degeneracies as a function of the energy. So at energy what's the ground state energy for the 3D harmonic oscillator? 3 halves h bar omega. It's three times the ground state energy for the single 1D harmonic oscillator. So 3/2 h bar omega. Yeah?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Good, the way we arrived that this was we found that the energy-- so the energy operator acting on the 3D wave function is what I get by taking the energy operator in 1D and acting on the wave function, and the energy operator in y acting on the wave function, the energy operator in z acting on the wave function, where the energy operator for each of those is the 1D harmonic oscillator with the same frequency. OK?

And then I separated. I said look, let the wave function, the 3D wave function, since I know this is separable and each separated part of the wave function satisfies the 1D harmonic oscillator equation, I know what the eigenfunctions of the 1D harmonic

oscillator energy eigenvalue problem are. They are the phi n's.

And so I can just take my 3D wave function. I can say the 3D wave function is going to be the separated form, the product of 1D wave function in x, a 1D wave function in y, and 1D wave function in z. Cool?

So now let's look back at what let's think about what happens when I take the energy operator and I act on that [INAUDIBLE]. When I take the 3D energy operator, which is the sum of the three 1D energy operators, harmonic oscillator energy operators. When thinking I'm going to act on this guy, the first one, which only knows about the x direction, sees the y and z parts as constants. And it's a phi x, and what does it give us back? E on phi x is just h bar omega n plus one half. Ditto for this guy. And then the energy operator in 3D is the sum of the three 1D energy operators. So that tells us the energy is the sum of the three energies. Is that cool? OK good. Other questions? Yeah?

AUDIENCE: Is the number of degeneracies essentially [INAUDIBLE] of number theory.

PROFESSOR: Ask me that after class. So let's look at the degeneracies as a function of the energy. So at the lowest possible energy, 3/2 omega, what states can I possibly have? I'm going to label the states by the three numbers, l, m, and n. So this is just the ground state 0 0 0. So is that degenerate? No, because there's just the one state.

What about at the next level? What's the next allowed energy?

AUDIENCE: 5/2.

PROFESSOR: 5/2. OK, 5/2. So at 5/2 what states do we have? Well, we have 1 0 0. But we also have 0, 1, 0. And we also 0 0 1. Aha, this is looking good. What does this correspond to physically? This says you have excitation, so you've got a node in the x direction. But your Gaussian in the y and z directions. This one says your Gaussian in the x direction. You have a node in the y direction as a function of y, because it's phi 1 of y. And you're Gaussian in the z directions. And this one says you have 1 excitation in the z direction. So they sound sort of rotated from each

other. That sounds promising.

But in particular, what we just discovered sort of by construction is that there can be degeneracies among bound states in 3D. This was not possible in 1D, but it is possible in 3D, which is cool.

But we've actually learned more. What's the form of the degeneracies? So here it looks like they're just rotations of each other. You call this x, I call it y, someone else calls it z. These can't possibly look different functions because they're just rotations of each other.

However, things get a little more messy when you write, well, what's the next level? What's the next energy after 5/2 h bar omega? 7/2 h bar omega, exactly. 7/2. And that's not so bad. So for that one, we get 2 0 0, 0 2 0, 0 0 2. But is that it?

AUDIENCE: No.

PROFESSOR: What else do we get? 1 1 0, 0 1 1, 1 0 1. So first off, let's look at the number. The degeneracy number here-- I'll call this d sub 0-- the degeneracy of the ground state is 1, OK? The degeneracy-- and in fact, I'm going to write this as a table-- the degeneracy as level n. So for d0 is equal to 1. d1 is equal to 3. And d2 is equal to 6.

Now it's less clear here what's going on, because is this just this guy relabelled? No. So this is weird, because we already said that the reason we expect that there might be degeneracy, is because of rotational symmetry. The system is rotationally invariant. The potential, which is the harmonic oscillator potential, doesn't care in what direction the radial displacement vector is pointing. It's rotationally symmetrical.

When we have symmetry-- on general grounds, when we have a symmetry, we expect to have degeneracies. But this are kind of weird, because these don't seem to be simple rotations of each other, and yet they're degenerate. So what's up with that? Question? Yeah?

AUDIENCE: [INAUDIBLE] Gaussian in certain directions?

PROFESSOR: Yeah, sure OK. So let me just explain what this notation means again. So by 1 0 0, what I mean is that the number l is equal to 1, the number m is equal to 0, n is equal to 0. That means that the wave function phi 3D is equal to phi 1 of x, phi 0 of y, phi 0 of z. But what's phi 0 of z? What's a Gaussian in the z direction? Phi 0 of y. That's the ground state in the y direction of the harmonic oscillator. It's Gaussian in the y direction.

If I wanted x, that's not the ground state. That's the excited state. And in particular, sort of being a Gaussian it goes through 0. It has a node. So this wave function is not rotationally invariant. It as a node in the x direction, but no nodes and y and z direction. And similarly for these guys. Did that answer your questions? Great.

OK, so we have these degeneracies, and they beg an explanation. And if you look at the next level, it turns out that d3-- and you can do this quickly on a scrap of paper-- d3 is 10, OK? And they go on. And if you keep writing this list out, I guess it goes up-- what's the next one? 15. 21, yeah.

So this has a simple mathematical structure, and you can very quickly convince yourself of the form of this degeneracy. dn is n n plus 1 over 2. So let's just make sure that works. 1 1 plus 1 over 2. Sorry I should really call this 1. n plus 1, n plus 2 if I count from 0. So for 0, this is going to give us 1 times 2 over 2. That's 1. That works. So for 1 that gives 2 times 3 over 2, which is 3, and so on and so forth.

So where did this come from? This is something we're going to have to answer. Why that degeneracy? That seems important. Why is it that number? Why do we have that much degeneracy?

But the thing I really want to emphasize at this point is that there's an absolutely essential deep connection between symmetries and degeneracies. If we didn't have symmetry, we wouldn't have degeneracy, and we can see that very easily here. Imagine that this potential was not exactly symmetric. Imagine we made it slightly different by adding a little bit of extra frequency to z direction. Make the z frequency slightly different. Plus m omega tilde squared upon 2 z squared, where omega tilde

is not equal to omega 0. OK?

The system is still separable, but this guy has frequency omega 0. The x part has omega 0. This has omega 0. But this has omega tilde. OK? And so exactly the same argument is going to go through, but the energy now is going to have a different form. The energy is going to have h bar omega-- h bar omega 0 times l plus m plus 1. But from the z part it's going to have plus h bar omega tilde times n plus 1/2.

And now these degeneracies are going to be broken, because this state will not have the same energy as these two. Everyone see that? When you have symmetry, you get to degeneracy. When you don't have a symmetry, you do not get degeneracy.

This connection is extremely important, because it allows you to do two things. It allows you to first not solve things you don't need to solve for. If you know there's a symmetry, solve it once and then compute the degeneracy and you're done.

On the other hand, if you have a system and you see just manifestly you measure the energies, and you measure that the energies are degenerate, you know there's a symmetry protecting those degeneracies. You actually can't be 100% confident, because I didn't prove that these are related to each other, but you should be highly suspicious. And in fact, this is an incredibly powerful tool in building models of physical systems. If you see a degeneracy or an approximate degeneracy, you can exploit that to learn things about the underlying system. Yeah?

AUDIENCE: So we just add the different omega to each omega [INAUDIBLE] number there is still a possibility to get a degeneracy.

PROFESSOR: Exactly. So it's possible for these omegas to be specially tuned so that rational combinations of them give you a degeneracy. But it's extraordinarily unlikely for that to happen accidentally, because they have to be rationally related to each other, and the rationals are a set of measures 0 in the reels.

So if you just randomly pick some frequencies, they'll be totally incommensurate,

and you'll never get a degeneracy. So it is possible to have an accidental degeneracy. Whoops, just pure coincidence. But it's extraordinarily unlikely. And as you'll see when you get to perturbation theory, it's more than unlikely. It's almost impossible. So it's very rare that you get accidental degeneracy. It happens, but it's rare. Other questions? OK, nothing?

OK so here we're now going to launch into-- so this leads us into a very simple question. At the end of the day, the degeneracies that we see for the 3D free particle, which is a whole sphere's worth of degeneracy, and the degeneracy we see for the 3D harmonic oscillator, the bound states, which is discrete, but with more and more degeneracy the higher and higher energy you go. Those we're blaming, at the moment, on a symmetry, on rotational symmetry, rotational invariance.

So it seems wise to study rotations, to study rotational invariance and rotational transformations in the first place. In the first part of the course, in 1D quantum mechanics, we got an awful lot of juice out of studying translations. And the generator of translations was momentum. So we're going to do the same thing now. We're going to study rotations and the generators of rotations, which are the angular momentum operators, and that's going to occupy us for the rest of today and Thursday. Yeah?

AUDIENCE: So your rotational symmetry will explain a factor of three in your degeneracy, right? But what's the symmetry that explains the way this grows. Because this very clearly appears there's 1 up to a factor of three. And then there's 2 up to a factor of three. And there's even more that's not even a multiple of three.

PROFESSOR: Right, actually so here's a very tempting bit of intuition. Very tempting bit of intuition is going to say the following. Look, rotational invariance, there's x, there's y, and there's z. It's going to explain rotations amongst those three. So that could only possibly give you a factor of three. But it's important to keep in mind that that's not correct intuition. It's tempting intuition, but it's not correct. And an easy way to see that its not correct is that for the free particle, there is a continuum, a whole sphere's

worth of degenerate states in any energy. And all of those are related to each other by simple rotation of the k vector, of the wave vector, right?

So the rotational symmetry is giving us a lot more than a factor of three. And in fact, as we'll see, it's going to explain exactly the n plus 1 n plus 2 over 2. OK, so with that motivation let's start talking about angular momentum.

So I found this topic to be not obviously the most powerful or interesting thing in the world when I first studied it. And my professor was like no, no, no. This is the deepest thing. And recently I had a fun conversation with one of my colleagues, Frank Wilczeck, who said yeah, in intro to quantum mechanics the single most interesting thing is the angular momentum and the addition of angular momentum. And something has happened to me in the intervening 20 years that I totally agree with him. So I will attempt to convey to you the awesomeness of this. But you have to buy in a little. So work with me in the math at the beginning of this, and it has a great payoff.

OK so the question is, what is the operator. So we're going to talk about angular momentum now. And I want to start with the following question. In the same sense as we started out by asking what represents position and momentum, linear momentum, in quantum mechanics, what represents what operator by our first, second, or third postulate-- I don't even remember the order now. What operator represents angular momentum in quantum mechanics?

And let's start by remembering what angular momentum is in classical mechanics. So L in classical mechanics is r cross p. In classical mechanics. So let's just try this. Let's construct that operator. This is not the world's most beautiful way of deriving this, but let's just write down natural guess. For in quantum mechanics what's the operator we want? Well, we want a vectors worth of operators, because angular momentum is a vector. It's a vector of operators, three operators. And I'm going to write these as r vector the operators x, y, z cross p the vector of momentum operators.

So at this point, you should really worry, because do r and p commute? Not so

much, right? However, the situation is better than it first appears. Let's write this out in terms of components. So this is in components. And I'm going to work, for the moment, in Cartesian coordinates. So Lx is equal to? Lx is equal to? You all took mechanics. Lx is equal to?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Thank you. YPz minus ZPy. And that's the curl, the x component of the curl. And similarly, the x component-- so the way to remember this is that its cyclic. x, y, z. y, z, x. So z, p, y. PX minus XPz. And then we have z XPy minus YPx.

So we were worried here about maybe an ordering problem. Is there an ordering problem here? Does it matter if I write YPz or PZy?

AUDIENCE: No.

PROFESSOR: No, because they commute with each other. PZ is momentum for the z-coordinate, not the y-coordinate, and they commute with each other. So there's no ambiguity. It's perfectly well defined. So we're just going to take this to be the definition of the components of the angular momentum operator Lx, Ly, and Lz.

And just for fun, I want to write this out. So because we know that Px, Py, and Pz can be expressed in terms of derivatives or differential operators, we can write the same operator in Cartesian coordinates in the following way. So clearly we could write this as Y d dx i upon h bar-- or sorry, h bar upon i. And z h bar upon i d dy. So we could write that in Cartesian coordinates as a differential operator.

But we can also write this in spherical coordinates. I'm just going to take a quick side note just to write down what it is. If we did this spherical coordinates, it's particularly convenient to write-- let me just write Lz for the moment. This is equal to minus i h bar derivative with respect to phi, where the coordinates, [INAUDIBLE] coordinates and spherical coordinates in this class is theta is going to be the angle down from the vertical axis, from the z-axis. And phi is going to be the angle of a period 2pi that goes around the equator, OK? Just to give you a name. This is typically what physicists call them. This is typically not what mathematicians call them. This leads

to enormous confusion. I apologize for my field.

So here it is. I can also construct the operator associated with the square of the momentum. And why would we care about the square of the momentum? Well, that's what shows up in the Hamiltonian, that's what shows up in the energy.

So I can construct the operator for the square of the momentum and write it, and it takes a surprisingly simple form. When I see surprisingly simple, you might disagree with me, but if you actually do the derivation of this, it's much worse in between. 1 over sine theta d theta sine theta d theta plus 1 over sine squared theta d phi.

Couple quick things. What are the dimensions of angular momentum? Length and momentum. What else has units of angular momentum?

AUDIENCE: h bar.

PROFESSOR: Solid. h bar, dimensionless. Angular momentum squared, angular momentum squared. OK, great. So that's going to be very convenient. h bars are just going to float around willy nilly.

OK, so suppose I ask you the following-- bless you. Suppose I ask you the following questions. I say look, here are the operators of angular momentum. This is Lz. We could have written down the same expression for Lx, and a Ly, and L squared. What are the eigenfunctions of these operators? Suppose I ask you this question. You all know how to answer this question.

You take these operators-- so for example, if I ask you what are the eigenfunctions of Lz? Well, that's not so bad, right? The eigenfunction of Lz is something where Lz on phi-- I'll call little m-- is equal to minus I h bar d d theta phi sub m. But I want the eigenvalue, so I'll call this h bar times some number. Let's call it m, because Lz is an angular momentum. It carries units of h bar, and its h bar times some number which is dimensional, so we'll call it m.

And we all know the solution of this equation. The derivative is equal to a constant times-- we can lose the h bar. We get a minus i, so we pick up an i. So therefore phi

sub m is equal to some constant times e to the im phi.

What can we say about m? Well, heres an important thing-- oh shoot. I'm using phi in so many different ways here. Let's call this not phi, because it's going to confuse the heck out of us. Let's call it Y. So we'll call it Y sub m. Why not? It's not my joke. This goes back to a bunch of-- yeah well, it goes way back.

So phi is the variable. Y is the deciding eigenfunction of Lz, and that's great. But what can you say about m? Well phi is the variable around the equator is periodic with period 2pi. And our wave function had better be single valued. So what does that tell you about m? Well, under phi goes to phi plus 2pi. This shifts by im 2pi. And that's only one to make a single valued if m is an integer. So m has to be an integer. m is an integer.

Now we did that for Lz. We found the eigenfunctions of Lz. What about finding the eigenfunctions L squared? Exactly the same thing. We're going to solve the eigenvalue equation, but it's going to be horrible, horrible to find these functions, right? Because look at this. 1 over sine squared d d phi. And then 1 over sine d theta sine d theta. This is not going to be a fun thing to do. So we could just brute force this, but let's not. Let's all agree that that's probably a bad idea. Let's find a better way to construct the eigenfunctions of the angular momentum operators.

So let's do it. So we ran into a situation like this before when we dealt with the harmonic oscillator. There was a differential equation that we wanted to solve. And OK, this one isn't nearly as bad, not nearly as bad as that one would have been. But still it was more useful to work with operator methods.

So let's take a hint from that and work with operator methods. So now we need to study the operators of angular momentum. So let's study them in a little more detail. So something you're going to show in your problem set is the following. The commutator of Lx with Ly takes a really simple form. This is equal to i h bar-- let's just do this out. Let's do this commutator. We're OK.

So Lx with Ly, this is equal to the commutator of YPz minus ZPy with Ly ZPx minus

XPz. So let's look at these term by term. So the first one is YPz ZPx. YPz ZPx. That's a Px. Sorry, ZPy. ZPx, this term. X, good. That's the first commutator.

The second commutator, it can be YPz an XPz minus YPz XPz. And then these commutators, the next two, are going to be ZPy with ZPx. And finally ZPy. And that's minus and this is a plus. ZPy and XPz.

OK, so let's look at these. These look kind of scary at first. But in this one notice the following. This is XPz ZPx minus ZPx YPz. But what you say about Y with all these other operators? Y commutes with all of them. So in each term I could just pull Y all the way out to one side. Yeah? So I could just pull out this Y. So for the first term, I'm going to write this as Y commutator PZ with ZPx.

And let me just do that explicitly. There's no reason to. So this is YPz ZPx minus ZPx YPz. And I can pull the Y out front, because this commutes with Px and with Z. So I can make this Y times PZ ZPx minus ZPx Pz. But now note that I can do exactly the same thing with the Px. Px commutes with z. Px commutes with Pz. And it commutes with y.

So I can pull the px from each term out. Px Y. And now I lose the Px. I lose the Px. But now this is looking good. This is Px times Y. And Pz minus ZPz PZz minus ZPz. This is also known as PXy times commutator of PZ with Z. And what is this equal to? Let's get our signs right. This is Px with Y time-- OK, we all agree that this is going to have an h bar in it. Let's write units. There's going to be an i. And is it a plus or minus? Minus. OK, good.

So we get minus h bar XPy. So this term is going to give us minus i h bar Py XPx times Y. And let's look at this term. OK, this is Y, and Y commutes with PZx and PZ, right? So I could just pull out the y. And x commutes with everything, so I can pull out the x. And I'm left with the commutator of Pz with Pz. What's the commutator of Pz with Pz?

AUDIENCE: 0.

PROFESSOR: 0. This term gives me a 0. Similarly here, Py commutes with everything. Z ZPx. Px

commutes with everything. Z ZPy. So I can pull out the Px Py, and I get Z commutator Z, and what's that?

AUDIENCE: 0.

PROFESSOR: 0. So this gives me 0. And now this term, ZPy XPz, the only two things that don't commute with each other are the Z and the Pz. The Py and the X I pull out, so I get it a term that's XPy and the commutator of Z with Pz. And what is that going to give me? PyX i h bar.

Aha, look at what we got. This is equal to i h bar times-- did I screw up the signs? XPy minus YPz. Oh sorry, Px. And what is this equal to?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yeah, i h bar Lz. And more generally, as you'll show on the problem set, you get the following commutators. Once you've done this once, you can do the rest very easily. Lx Ly is-- so Lx with Ly is i h bar Lz. And then the rest can be got from cyclic rotations Ly with Lz is i h bar Lx and Lz with Lx is i h bar Ly.

And now here's a fancier one. This is less obvious, but exactly the same machinations will give you this result, and you'll do this again on the problems set. If I take L squared, k and L squared here is going to be L squared, I just mean Lx squared plus Ly squared plus Lz squared. This is the norm squared of the vector, the operator form. L squared with Lz-- or sorry, with Lx. Yeah, fine. Lx is equal to 0. So Lx commutes with the magnitude L squared.

Similarly L squared, now just by rotational invariance, if Lx commutes with it, Ly and Lz had better also commute with it, because of who you to see what's Lx. And L squared with Lz must be equal to 0. Yeah? Everyone cool with that?

And the thing is we've just-- and I'm going to put big flags around this. We've just learned a tremendous amount about the eigenfunctions of the angular momentum operators. Why? Let's leave that up. So what have we just learned about the eigenfunctions of the angular momentum operators Lx, Ly, Lz, and L squared?

Anyone? We just learned something totally awesome about them. Anyone?

So can you find simultaneous eigenfunctions of Lx and Ly?

AUDIENCE: No.

PROFESSOR: Not so much. They don't commute to 0. What about Ly and Lz? Nope. Lz Lx, nope. So you cannot find simultaneous eigenfunctions of Lx and Ly. What about Lx and L squared? Yes. So we can find simultaneously eigenfunctions of L squared and Lx. OK, what about can we find simultaneous eigenfunctions of L squared and Ly? Yep. Exactly. What about L squared, Ly, and Lz? Nope. No such luck.

And this leads us to the following idea. The idea is a complete set of commuting observables. And here's what this idea is meant to contain. You can always write down a lot of operators. So let me step back and ask a classical question. Classically, suppose I have a particle in three dimensions, a particle moving around in this room, non relativistic, familiar 801. I have a particle moving around in this room.

How much data must I specify to specify the configuration of this system? Well, I have to tell you where the particle is, and I have to tell you what its momentum is, right? So I have to tell you the three coordinates and the three momenta. If I give you five numbers, that's not enough, right? I need to give you six bits of data. On the other hand, if I give you seven numbers, like I give x, y, z, Px, Py, Pz, and e, that's over complete. Right? That was unnecessary.

So in classical mechanics, you can ask what data must you specify to completely specify the state of the system. And that's usually pretty easy. You specify the number coordinates and the number of momenta.

In quantum mechanics, we ask a slightly different question. We ask, in order to specify the state of a system, we say we want to specify which state it is. You can specify that by saying which superposition in a particular basis. So you pick a basis, and you specify which particular superposition, what are the eigenvalues of the operators you've diagonalized in that basis?

So another way to phrase this is for a 1D problem, say we have just a simple 1D problem. We have X and we have P. What is a complete set of commuting operators? Well, you have to have enough operators so that the eigenvalue specifies a state. So X would be-- is that enough? So if I take the operator X and I say look, my system is in the state with an eigenvalue X not of X, does that specify the state of the system? It tells you the particles are in a delta function state right here. Does that specify the state?

AUDIENCE: Yes.

PROFESSOR: Fantastic, it does. Now what if I tell you instead, oh it's in a state of definite P. Does that specify the state? Absolutely. Can I say it's in a state with definite X and definite P?

AUDIENCE: No.

PROFESSOR: No. These don't commute. So a complete set of commuting observables in this case would be either X or P, but not both. Yeah?

Now if we're in three dimensions, is x a complete set of commuting observables?

AUDIENCE: No.

PROFESSOR: No, because it's not enough. You tell me that it's X, that doesn't tell me what state it is because it could have y dependence or z dependence.

So in 3D, we take, for example, x, and y, and z. Or Px, and Py, and Pz. We could also pick z, and Px, and y. Are these complete? If I tell you I have definite position in z, definite position in y, and definite momentum in x? Does that completely specify my state?

AUDIENCE: Yes.

PROFESSOR: Yeah, totally unambiguously. e to the ikx delta of y delta of z totally fixes my function, completely specifies it. If I'd only picked two of these, it would not have

been complete. It would have told me, for example, e to the ikx delta of y, but it doesn't tell me how it depends on z, how the wave functions on z. And if I added another operator, for example, Py, this is no longer commuting. These two operators don't commute.

So a complete set of commuting observables can be thought of as the most operators you can write down that all commute with each other and the minimum number whose eigenvalues completely specify the state of the system. Cool? OK.

So with all that said, what is a complete set of commuting observables for the angular momentum system? Well, it can't be any two of Lx, Ly, and Lz. So let's just pick one. I'll call it Lz. I could've called it Lx. It doesn't matter. It's up to you what axis is what. I'll just call it Lz conventionally.

And then L squared also commutes, because L squared commutes with Lx. It commutes Ly and with Lz. So this actually forms a complete set of commuting observables for the angular momentum system. Complete set of commuting observables for angular momentum. So this idea will come up more in the future.

And here is going to be the key [INAUDIBLE]. So we'd like to use the following fact. We want to construct the eigenfunctions of our complete set of-- yeah, question?

AUDIENCE: Really quick can you explain how you got that L squared and Lx commute?

PROFESSOR: Yeah, I got it by knowing what you're going to write on your solution set. So this is on your problem set. So the way it goes-- so there are fancy ways of doing it, but the just direct way of doing, how do you construct these commutators? Is you know what the operators are. You know what L squared is. And you know that L squared is Lx squared plus Ly squared plus Lz squared. It's built in that fashion out of x and Py.

And then I literally just put in the definitions of Lx, Ly, and Lz into that expression for L squared and compute the commutator with Lx, again, using the definition in terms of Py and z. And then you just chug through the commutators. Yeah, it just works out. So it's not obvious from the way I just phrased it that it works out like that. Later

on you'll probably develop some intuition that it should be obvious. But for the moment, I'm just going to call it a brute force computation. And that's how you're going to do it on your problem set.

So let me tell you where we're going next. So the question I really want to deal with is what are the eigenfunctions? So this is where we're leaving off. What are the eigenfunctions of our complete set of commuting variable L squared and Lz. What are these guys? And we know that we could solve them by solving the differential equations using those operators, but that would be horrible. We'd like to do something a little smarter. We'd like to use the commutation relations and the algebra.

And here a really beautiful thing is going to happen. When you look at these commutation relations, one thing they're telling us is that we can't simultaneously have eigenfunctions of Lx and Ly. However, the way that Lx and Ly commute together is to form Lz. That gives us some information. That gives us some magic, some power.

And in particular, much like that moment in the harmonic oscillator I said, well look, we could write down these operators as a. Well look, we can write down these operators, which I'm going to call L plus and L minus. L plus is going to be equal to Lx plus i Ly. And L minus is going to be equal to Lx minus I Ly.

Now Lx, Ly, those are observables? What can you say about them as operators? What kind of operators are they since they're observables? Hermitian, exactly. So what's the Hermitian adjoint of Hermitian plus i Hermitian? Hermitian minus i Hermitian. OK, good. So L minus is the adjoint of L plus.

So this is just going to be a definition. Let's take these to be the definitions of these guys. If we take their commutator, something totally lovely happens. I'm not going to write all the commutators. I'm just going to write a couple.

The first is if I take L squared and I commute with L plus, well, L plus is Lx plus Ly, and we already know that L squared commutes with Lx and it commutes with Ly. So

L squared commutes with L plus. And similarly for L minus, it commutes with each term. Question? Oh, I'm sorry. Lx. Shoot, that was an x. It just didn't look like it. Lx minus i Ly.

So it commutes both L plus and L minus. But here's the real beauty of it. Lz with L plus is equal to-- well, let's just do dimensional analysis. L plus is Lx plus i Ly. We know that Lz with Lx is something like Ly. And Lz with Ly is something like Lx. With factors of i's an h bars.

And when you work out the commutator, which should only take you second, you get i h bar L plus. And similarly, when we construct Lz and L minus, we get minus h bar L minus. Are L plus an L minus Hermitian? No, they're each other's adjoints. Lz is Hermitian.

And look at this commutation relation. What does that tell you? From the first observation, if we have an energy or if we have an operator e and an operator a that commute in this fashion, then this tells you that the eigenfunctions of this operator are staggered in a the ladder spaced by h bar. The eigenvalues of Lz come in a ladder spaced by h bar. We can raise with L plus, and we can lower with L minus just like in the harmonic oscillator problem. And we'll exploit the rest of the-- we'll deduce the rest of the structure of the angular momentum operator eigenfunctions next time using this computation relation. See you next time.