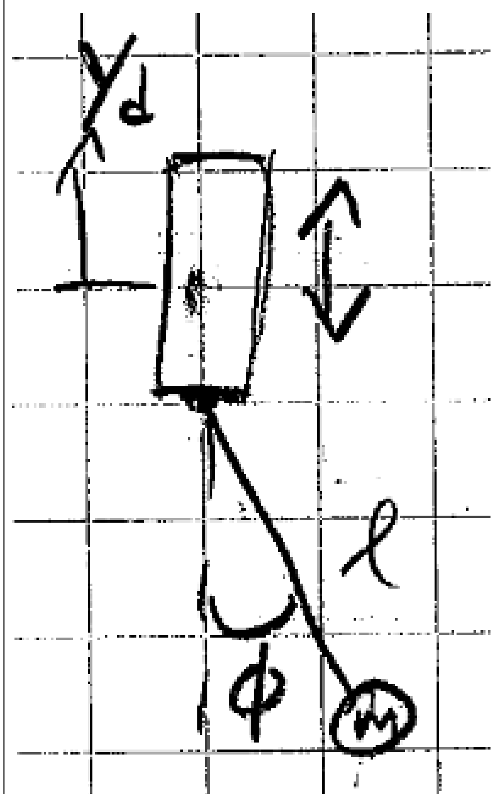
(14) Tricky Potentials

1 Kaptiza Example

Let's try a slightly different pendulum system this time for our next example.

Kapitza



$$T = \frac{1}{2}m\left(\dot{x}_m^2 + \dot{y}_m^2\right), \;\; U = mgy_m, \;\; D = \frac{1}{2}b\dot{\phi}^2$$

$$x_m = l\sin\phi \;\; , \;\; y_m = y_d - l\cos\phi$$

$$\dot{x}_m = l\cos\phi\dot{\phi} \;\; , \;\; \dot{y}_m = \dot{y}_d + l\sin\phi\dot{\phi}$$

The Lagrangian for this system, after dropping terms which contain only y_d, which is an explicit function of time, is

$$
\begin{aligned}
L &= \frac{1}{2}m\left(l^2\dot{\phi}^2 + 2l\sin\phi\,\dot{\phi}\dot{y}_d\right) + mgl\cos\phi \\
\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) &= ml\left(l\ddot{\phi} + \cos\phi\dot{\phi}\dot{y}_d + \sin\phi\ddot{y}_d\right) \\
\frac{\partial L}{\partial\phi} - \frac{\partial D}{\partial\dot{\phi}} &= ml\left(\cos\phi\dot{\phi}\dot{y}_d - g\sin\phi\right) - b\dot{\phi}
\end{aligned}
$$

which gives us the equation of motion.

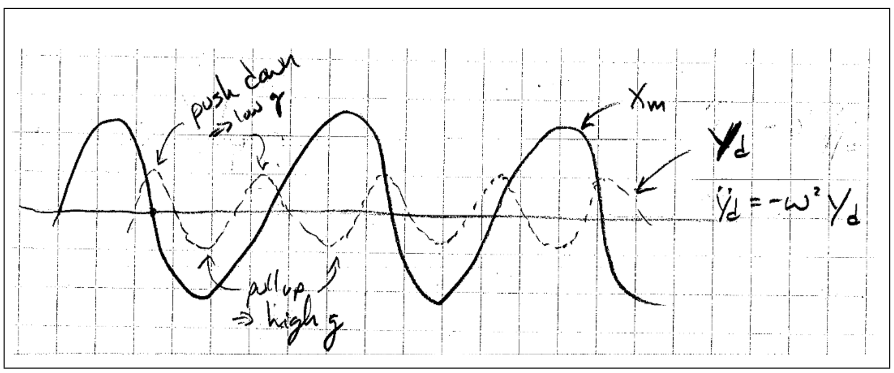
A vertically driven pendulum is a bit of a strange thing; it doesn't seem to work as a driver!

$$
l\ddot{\phi} + \frac{b}{lm}\dot{\phi} + (g + \ddot{y}_d)\sin\phi \;= 0
$$

where the damping term is $\frac{b}{lm}\dot{\phi}$.

Instead, the drive appears to modify gravity. This makes sense, due to the equivalence principle. Interestingly, this let's us explore parametric resonance...

Notice how the pendulum becomes excited with a drive at twice the resonance frequency.



We won't cover parametric resonance further, but LL27 does. Methods for understanding non-linear/anharmonic behavior are also covered in LL 28-29, but I found the math unenlightening, so I won't try to reproduce it here.

We also see strange behavior for a high frequency drive. Damping is not important for this, so let's operate with $b = 0$. We can understand this by noticing that the pendulum's motion consists of a high frequency part (at the drive frequency) and a low frequency part (swinging around).

$$\phi(t) \simeq \phi_1(t) + \phi_2(t)$$

where ϕ_1 corresponds to slow oscillations, and ϕ_2 to fast

$$l\ddot{\phi} + (g + \ddot{y}_d)\sin\phi = 0$$

$$l\left(\ddot{\phi}_1 + \ddot{\phi}_2\right) + (g + \ddot{y}_d)\sin(\phi_1 + \phi_2) = 0$$

assume $\phi_1 \sim$ const, and $\phi_2 \ll 1$

$$l\ddot{\phi}_2 + (g + \ddot{y}_d)(\sin\phi_1 + \cos\phi_1\phi_2) = 0$$

$$\begin{aligned} \text{for } y_d &= a_d\cos(\omega t), \\ \ddot{y}_d &= -a_d\omega^2\cos\omega t = -\omega^2 y_d \end{aligned}$$

Fast oscillation terms, to first order, are

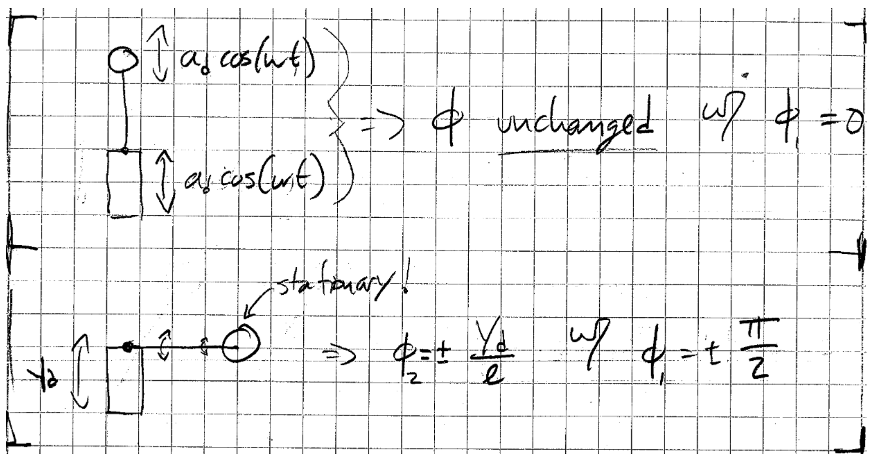
$$l\ddot{\phi}_2 + g\cos\phi_1\phi_2 = a_d\omega^2\sin\phi_1\cos\omega t$$

driven response:

$$\Rightarrow \phi_2 \simeq \frac{a_d\omega^2\sin\phi_1}{l(\omega_0^2\cos\phi_1 - \omega^2)}\cos\omega t \simeq -\sin\phi_1\frac{y_d}{l} \quad \text{for } \omega \gg \omega_0$$

Graphically this result is

High Frequency Drive



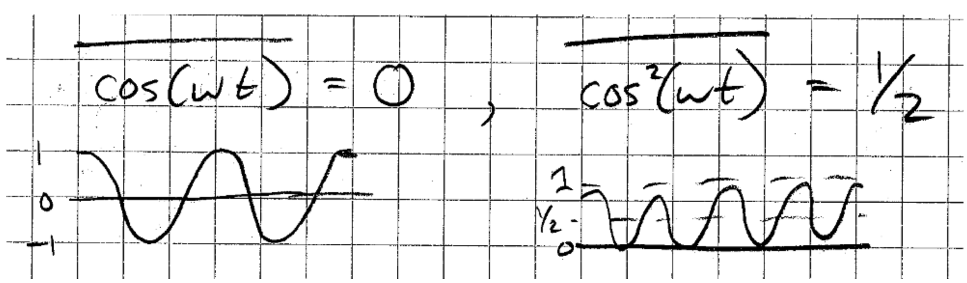
Returning to our equations of motion, but keeping $\phi_2 \ll 1$ and $\ddot{\phi}_2 = -\sin\phi_1 \frac{\ddot{y}_d}{l}$

$$l\ddot{\phi}_1 + \omega^2 \sin\phi_1 y_d + \left(g - \omega^2 y_d\right) \sin\phi_1 \left(1 - \cos\phi_1 \frac{y_d}{l}\right) = 0$$

$$l\ddot{\phi}_1 + g \sin\phi_1 + \frac{\sin 2\phi_1}{2l} \left(\omega^2 y_d^2 - g y_d\right) = 0$$

We are looking for the slow behavior, so let's average over the fast drive period. Since $y_d = a_d \cos(\omega t)$

Slow Behavior



$$\Rightarrow \ddot{\phi}_1 + \frac{g}{l}\sin\phi_1 + \frac{1}{4}\left(\frac{a_d\omega}{l}\right)^2 \sin(2\phi_1) = 0$$

If we are close to $\phi_1 \simeq \pi$ (pointing up)

$$\text{for } \phi_1 = \pi + \varepsilon \text{ with } \varepsilon \ll 1 \Rightarrow \sin\phi_1 \simeq -\varepsilon,\ \ \sin 2\phi_1 \simeq 2\varepsilon$$
$$\Rightarrow \ddot{\varepsilon} + \left(\tfrac{1}{2}\left(\tfrac{a_d\omega}{l}\right)^2 - \tfrac{g}{l}\right)\varepsilon = 0$$

$$\Rightarrow \quad \text{oscillator with } \ \omega_0^2 = \tfrac{1}{2}\left(\tfrac{a_d\omega}{l}\right)^2 - \tfrac{g}{l}$$
$$\text{stable if } a_d^2\omega^2 > gl$$

So, as we have seen, the Kapitza pendulum is stable around $\phi \sim \pi$ (i.e. inverted) given a sufficiently fast drive.

Generally, when treating motion in a rapidly oscillating field, we can define an effective potential

$$U_{eff} = U + \bar{T} = U(q_{\text{slow}}) + \tfrac{1}{2}m\overline{dotq}^2_{\text{fast}}$$
$$\text{where } q_{\text{fast}} \text{ is the fast part of } q(t) = q_{\text{slow}}(t) + q_{\text{fast}}(t)$$

for us, this would be

$$E = \tfrac{1}{2}ml^2\left(\dot{\phi}_1 + \dot{\phi}_2\right)^2 + mg\left(y_d - l\cos(\phi_1 + \phi_2)\right)$$
$$\text{average over fast oscillations } (\bar{y_d} = 0)$$
$$\Rightarrow \overline{E} = \underbrace{\frac{1}{2}m(l\dot{\phi}_1)^2}_{T} + \underbrace{\frac{1}{2}m\overline{(l\dot{\phi}_2)^2} - mgl\cos(\phi_1)}_{U_{\text{eff}}}$$

$$\begin{aligned} U_{eff} &= -mgl\cos\phi_1 + \frac{1}{2}m\overline{(-\sin\phi_1 \dot{y}_d)^2} \\ &= -mgl\cos\phi_1 + \frac{1}{2}m\sin^2\phi_1\left(\frac{1}{2}a_d\omega\right)^2 \end{aligned}$$

Note:

$$\frac{\partial U_{eff}}{\partial \phi} = mgl\sin\phi_1 + m\sin(2\phi)\left(\tfrac{1}{2}a_d\omega\right)^2$$

divide by ml to get our equation of motion for ϕ_1

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2 Tricky Potentials

Already in this course we have seen a few tricky potentials.

For central potentials, angular momentu gives us an **effective** potential for r.

Central

$$\begin{aligned} U_{eff}\left(\vec{q},\dot{\vec{q}}\right) = U_{eff}\left(r,\dot{\phi}\right) &= U(r) + \frac{1}{2}\mu\left(r\dot{\phi}\right)^2 \\ \Rightarrow U_{eff}(r, L_z) &= U(r) + \frac{L_z^2}{2\mu r^2} \end{aligned}$$

A homogeneous dissipative medium, which converts kinetic energy of the macroscopic to kinetic energy of the microscopic (i.e. heat) can also be treated as a velocity dependent potential

Dissipative

$$\begin{aligned} U_{diss}(q,\dot{q}) &= U_{con}(q) - \int D(\dot{q})\,dt \\ \Rightarrow \frac{d}{dt}\left(\frac{\partial L_{con}}{\partial \dot{q}}\right) &= \frac{\partial L_{con}}{\partial q} - \frac{\partial D}{\partial \dot{q}} \end{aligned}$$

And last time we saw another sort of tricky potential for rapidly oscillating fields or drives

Rapid Drive

$$U_{eff}(q, \dot{q}) = U(q) + \frac{1}{2}m\overline{\dot{q}^2_{fast}}$$

$$\text{for } q(t) = q_{slow}(t) + q_{fast}(t)$$

These are all scalars. Need to note that U_{eff} comes from including the same T, which can be written as $T(q)$, in U to get $U_{eff} = U(q) + T(q)$.

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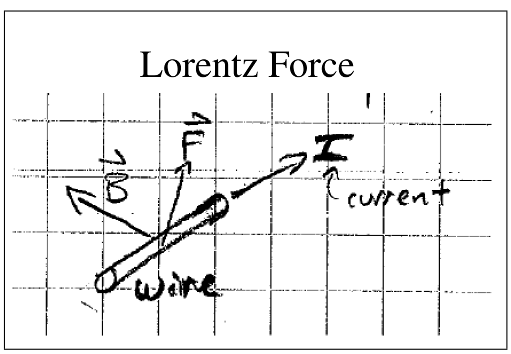
3 Lorentz Force

Today we encounter another tricky potential, this one from 8.02. The force on a charged particle moving in E and B fields is (as you may recall)

Lorentz force on particle with charge e (not q!)

$$\vec{F} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

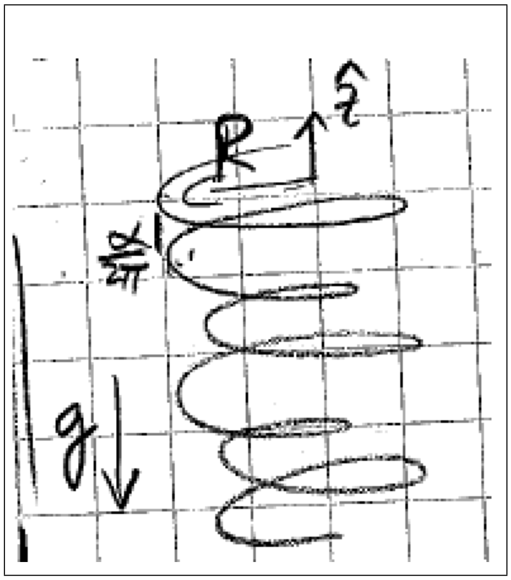
where $\vec{E}$ is the electric field and $\vec{B}$ is the magnetic field.



Many everyday objects use electric motors and/or generators, all of which depend on the Lorentz force.

I'm not going to use q here to avoid confusion. e is some charge and $\vec{r}$ is the Cartesian Coordinate of that charge.

Let's say I have a small test charge constrained to move along a wire in an external B-field. The B-field can be along $\hat{z}$, and the wire can be a helical coil, like a spring.



$$\begin{aligned} x &= R\cos\phi \qquad \dot{x} = -R\sin\phi\dot{\phi} \\ y &= R\sin\phi \qquad \dot{y} = R\cos\phi\dot{\phi} \\ z &= \alpha\phi \qquad \dot{z} = \alpha\dot{\phi} \end{aligned}$$

How does the charge move? Ideas? Let's find out... we need a potential for the Lorentz force

$$\begin{aligned} U_L\left(\vec{r}, \dot{\vec{r}}\right) &= e\left(\Phi - \vec{A}\cdot\dot{\vec{r}}\right) \\ \vec{E} &= -\nabla\Phi\left(\vec{r}, t\right) - \frac{\partial}{\partial t}\vec{A}\left(\vec{r}, t\right) \\ \vec{B} &= \nabla\times\vec{A} \end{aligned}$$

$$\Phi \equiv \text{electric scalar potential}$$
$$\vec{A} \equiv \text{magnetic vector potential}$$
$$\text{NB: for constant, uniform B-field, } \vec{A} = \frac{1}{2}\vec{r} \times \vec{B}$$

So, for our test charge we have

$$\begin{aligned} \vec{B} &= B\hat{z} \quad \Rightarrow \quad \vec{A} = \frac{B}{2}\{-y, x, 0\} \\ U &= mgz - \vec{A}\cdot\dot{\vec{r}} = mgz + \frac{eB}{2}(y\dot{x} - x\dot{y}) \\ &= mg\alpha\phi + \frac{eBR^2}{2}\left(-\sin^2\phi - \cos^2\phi\right)\dot{\phi} \end{aligned}$$

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \tfrac{1}{2}m\left(R^2 + \alpha^2\right)\dot{\phi}^2$$
$$\Rightarrow L = \tfrac{1}{2}m\left(R^2 + \alpha^2\right)\dot{\phi}^2 - mg\alpha\phi + \underbrace{\frac{eBR^2}{2}\dot{\phi}}_{drop!}$$

The dynamics are unchanged by the B-field! Why?

$$F \cdot v = \left(\vec{v} \times \vec{B}\right)\cdot\vec{v} = 0 \Rightarrow \text{no work done}$$

This will be true for any 1-D motion, so let's try 2-D...

How about a charge free to move in the y-z plane with a B-field in the $\hat{x}$ direction?

$$\vec{B} = B\hat{x} \quad \Rightarrow \quad \vec{A} = \tfrac{B}{2}\{0, -z, y\}$$
$$\Rightarrow L = \tfrac{1}{2}m\left(\dot{y}^2 + \dot{z}^2\right) + \tfrac{eB}{2}\left(y\dot{z} - z\dot{y}\right) - mgz$$

$$F_y = \frac{\partial L}{\partial y} = \frac{eB}{2}\dot{z}, \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} - \frac{eB}{2}z$$
$$\dot{p}_y = F_y \Rightarrow m\ddot{y} - \frac{eB}{2}\dot{z} = \frac{eB}{2}\dot{z}$$

$$F_z = \frac{-eB}{2}\dot{y} - mg, \quad p_z = m\dot{z} + \frac{eB}{2}y$$
$$\Rightarrow \ddot{y} = \beta\dot{z}, \quad \ddot{z} = -g - \beta\dot{y}$$
$$\text{with } \beta = \frac{eB}{m}$$

These equations of motion are simple enough to solve, and the solution is interesting...let's see what happens for a particle that starts at rest.

If you ignore g, you might guess that since the Lorentz force is $\perp$ to $\vec{v}$, the trajectory must be a circle.

$$\begin{aligned} y(t) &= a\sin\omega t \Rightarrow \dot{y} = -\omega z, \;\; \ddot{y} = -\omega^2 y \\ z(t) &= -a\cos\omega t \Rightarrow \dot{z} = \omega y, \;\; \ddot{z} = -\omega^2 z \end{aligned}$$

Another way to see that the trajectory must be a circle is to notice that

$$\dddot{z} = -\beta\ddot{y} = -\beta^2\dot{z} = -\omega^2\dot{z}$$
$$\text{with } \beta = \omega$$

which is the time derivative of equation of motion for a harmonic oscillator with frequency β.

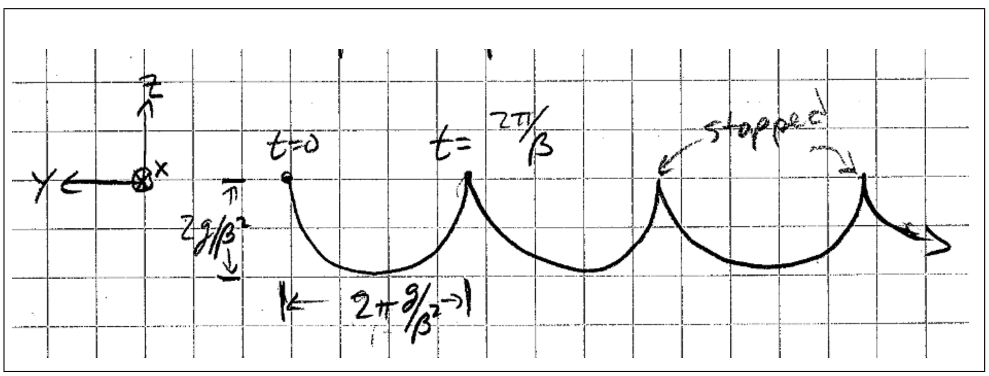
Comparing with our equations of motion suggests that $\omega = \beta$, but we have this pesky gravity... no problem, add $-\frac{g}{\beta}t$ to $y(t)$. This doesn't change $\ddot{y}$.

$$
\begin{aligned}
y(t) &= a \sin \beta t - \frac{g}{\beta} t \\
\dot{y} &= a\beta \cos \beta t - \frac{g}{\beta} \\
\dot{z} &= a\beta \sin \beta t
\end{aligned}
$$

To start at rest, we need

$$
\dot{y}\,(t=0) = 0 \;\;\Rightarrow\;\; a = -\frac{g}{\beta^2}
$$

$$
\begin{aligned}
y(t) &= -\frac{g}{\beta^2}\,(\sin \beta t + \beta t) \\
z(t) &= \frac{g}{\beta^2} \cos \beta t
\end{aligned}
$$



So this particle doesn't **fall**, it moves **sideways** with average velocity $\frac{g}{\beta}$ (for general initial conditions, you get $\sin$ and $\cos$ components for both y and z).

We can quickly relate this result to particle accelerators, though our non-relativistic physics is clearly inadequate to get a good answer...

For a particle moving in a plane $\perp$ to gravity, and with our B-field pointing up, we can reuse the previous result

$$\text{for } \vec{B} = B\hat{z} \qquad \ddot{x} = \beta\dot{y} \;,\; \ddot{y} = \beta\dot{x}$$
$$x(t) = a\sin\beta t \Rightarrow \dot{x} = a\beta\cos\beta t$$
$$y(t) = -a\cos\beta t \Rightarrow \dot{y} = a\beta\sin\beta t$$

So if we start a **proton** with an initial velocity c in the $\hat{x}$ direction...

$$\text{for } \vec{v_0} = c\hat{x} \;\;\Rightarrow\;\; a = \frac{c}{\beta} = \frac{m_p c}{eB}$$
$$\text{for } B \simeq 8 \text{ Tesla} \;\;\Rightarrow\;\; a = 0.4 \text{ m}$$

with $c = 3 \times 10^8 \text{m/s}$, $m_p = 1.7 \times 10^{-27}$ kg and $e = 1.6 \times 10^{-19}$ C.
But CERN has a radius of 4.5 km!
If we replace m_p with the relativistic mass $m_r = E_p/c^2$ we get the right answer:

$$m_r = \frac{E_p}{c^2} \sim 10^4 m_p \Rightarrow a_r = \frac{E_p}{eBc} \sim 4.2 \text{ km}$$
$$\text{for } E_p \approx 10 \text{ Tev} \approx 1.6 \times 10^{-6} \text{ J}$$

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4 Gauge Invariance

We have some freedom in choosing the magnetic vector potential $\vec{A}$

Gauge Transformation

$$\begin{aligned}
\Phi' &= \Phi - \frac{\partial}{\partial t} f \qquad \vec{A}' = \vec{A} + \nabla f \\
\vec{B}' &= \nabla \times \vec{A}' = \nabla \times \left(\vec{A} + \nabla f\right) = \vec{B} + \underbrace{\nabla \times (\nabla f)}_{\text{this is 0}} \\
\vec{E}' &= -\nabla \Phi' - \frac{\partial}{\partial t} \vec{A}' = \\
&= -\nabla \left(\Phi - \frac{\partial}{\partial t} f\right) - \frac{\partial}{\partial t}\left(\vec{A} + \nabla f\right) = \vec{E}
\end{aligned}$$

What about our Lagrangian?

Gauge Invariant Equation of Motion

$$\begin{aligned}
L' = \Phi - e\left(\Phi' - \vec{A}' \cdot \vec{r}\right) &= L + e\left(\frac{\partial}{\partial t} f + \dot{\vec{r}} \cdot \frac{\partial}{\partial \vec{r}} f\right) \\
&= L + e \frac{d}{dt} f
\end{aligned}$$

For Interesting physics associated with magnetic vector potential, google Aharnov-Bohm effect.