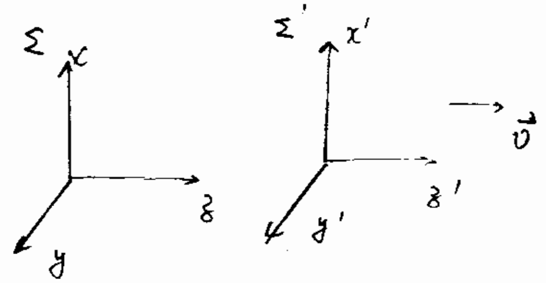
1. Change moving at constant velocity.



(a) Set the coordinate frame in which the charge is rest to be Σ'

And Let $\vec{v}$ along $\hat{z}$ direction

Then in the Lorentz gauge $\nabla \cdot \vec{A} + \frac{1}{c}\frac{\partial}{\partial t}\phi = 0$,

$(\phi, \vec{A})$ is a four-vector, so we have the Lorentz Transformation

$$\begin{cases} \phi = \gamma(\phi' + \vec{\beta}\cdot\vec{A}') \\ \vec{A} = \gamma(\vec{A}' + \vec{\beta}\phi') \end{cases} \quad \text{where } \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \quad \vec{\beta} = \vec{v}/c \qquad (1.1)$$

And we can easily obtain $(\phi', \vec{A}')$ in Σ'

$$\begin{cases} \phi' = \frac{e}{r'} \\ \vec{A}' = 0 \end{cases} \qquad (1.2)$$

(1.2) & (1.1) $\rightarrow$

$$\begin{cases} \phi = \gamma\phi' = \frac{\gamma e}{\sqrt{x'^2 + y'^2 + z'^2}} \\ \vec{A} = \gamma\vec{\beta}\phi' = \vec{\beta}\phi = \frac{\vec{v}}{c}\phi \end{cases} \qquad (1.3)$$

And we also have Lorentz Transformation for $(ct, \vec{r})$

$$\begin{cases} x' = x \\ y' = y \\ z' = \gamma(z - \beta ct) \\ ct' = \gamma(ct - \beta z) \end{cases} \qquad (1.$$

(1.4) into (1.3) $\rightarrow$

$$\phi = \frac{\gamma e}{\sqrt{x^2+y^2+\gamma^2(z-\beta ct)^2}}$$

$$= \frac{e}{\sqrt{(z-\beta ct)^2 + \frac{1}{\gamma^2}(x^2+y^2)}}$$

$$= \frac{e}{\sqrt{(z-\beta ct)^2 + (1-\frac{v^2}{c^2})(x^2+y^2)}}$$

$$= \frac{e}{\sqrt{[(z-\beta ct)^2 + x^2+y^2] - \frac{v^2}{c^2}(x^2+y^2)}}$$

$$= \frac{e}{\sqrt{(\vec{r}-\vec{v}t)^2 - \frac{v^2}{c^2}\vec{r}_\perp^{\,2}}}$$ where $\begin{cases} \vec{r}_\perp^{\,2} = x^2+y^2 \\ (\vec{r}-\vec{v}t)^2 = x^2+y^2+(z-vt)^2 \end{cases}$

Hence, we have shown in the Lorentz Gauge

$$\begin{cases} \phi = \dfrac{e}{\sqrt{(\vec{r}-\vec{v}t)^2 - \frac{v^2}{c^2}\vec{r}_\perp^{\,2}}} \\ \vec{A} = \dfrac{\vec{v}}{c}\phi \end{cases} \qquad (1.5)$$

(b) electric field

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}$$

$$= -\nabla\frac{e}{\sqrt{(1-\frac{v^2}{c^2})(x^2+y^2)+(z-\beta ct)^2}} - \frac{\vec{v}}{c^2}\frac{\partial}{\partial t}\frac{e}{\sqrt{(z-\beta ct)^2+(1-\frac{v^2}{c^2})(x^2+y^2)}}$$

$$= e \frac{(1-\frac{v^2}{c^2})x\hat{e}_x + (1-\frac{v^2}{c^2})y\hat{e}_y + (z-\beta ct)\hat{e}_z}{\left[(1-\frac{v^2}{c^2})(x^2+y^2)+(z-\beta ct)^2\right]^{\frac{3}{2}}} - \beta\frac{\vec{v}}{c}\frac{(z-\beta ct)\,e}{\left[(1-\frac{v^2}{c^2})(x^2+y^2)+(z-\beta ct)^2\right]^{\frac{3}{2}}}$$

$$= \frac{e(1-\frac{v^2}{c^2})\left[x\hat{e}_x + y\hat{e}_y + (z-\beta ct)\hat{e}_z\right]}{\left[(1-\frac{v^2}{c^2})(x^2+y^2)+(z-\beta ct)^2\right]^{\frac{3}{2}}}$$

$$= e(1-\frac{v^2}{c^2})\frac{\vec{r}-\vec{v}t}{\left[(\vec{r}-\vec{v}t)^2-\frac{v^2}{c^2}\vec{r}_\perp^{\,2}\right]^{\frac{3}{2}}} \qquad \text{where } \vec{v} = \beta c\,\hat{e}_z$$

The magnetic field

$$\vec{B} = \nabla\times\vec{A}$$

$$= \nabla\times\left[\frac{\vec{v}}{c}\frac{e}{\sqrt{(z-\beta ct)^2+(1-\frac{v^2}{c^2})(x^2+y^2)}}\right]$$

$$= \frac{e}{c}\left[(1-\frac{v^2}{c^2})(x^2+y^2)+(z-\beta ct)^2\right]^{-\frac{3}{2}}\Big\{\left[v_y(z-\beta ct)-v_z y(1-\frac{v^2}{c^2})\right]\hat{e}_x$$

$$+\left[v_z x(1-\frac{v^2}{c^2}) - v_x(z-\beta ct)\right]\hat{e}_y + \left[v_x y(1-\frac{v^2}{c^2}) - v_y x(1-\frac{v^2}{c^2})\right]\hat{e}_z\Big\}.$$

$\vec{v} = (v_x\ v_y\ v_z)$
$= (0\ 0\ v)$

$$= \frac{e}{c}\left[(1-\frac{v^2}{c^2})(x^2+y^2)+(z-\beta ct)^2\right]^{-\frac{3}{2}}\left\{-vy(1-\frac{v^2}{c^2})\hat{e}_x + vx(1-\frac{v^2}{c^2})\hat{e}_y\right\}$$

adding this term because $\vec{v}\times\vec{v}=0$

$$= \frac{v\hat{e}_z}{c}\times\frac{e(1-\frac{v^2}{c^2})(\vec{r}-\vec{v}t)}{\left[(1-\frac{v^2}{c^2})(x^2+y^2)+(z-\beta ct)^2\right]^{\frac{3}{2}}}$$

$$= \frac{\vec{v}}{c}\times\vec{E}$$

2 Potentials and fields of an arbitrarily moving charge.

(a) ① From the retarded potential Eq. (31.49)

$$\phi(\vec{r},t) = \int (d\vec{r}\,')\,dt' \; \frac{\delta\left(\frac{1}{c}|\vec{r}-\vec{r}\,'| - (t-t')\right)}{|\vec{r}-\vec{r}\,'|} \; \rho(\vec{r}\,',t')$$

$$= \int (d\vec{r}\,')\,dt' \; \frac{\delta\left(\frac{1}{c}|\vec{r}-\vec{r}\,'| - (t-t')\right)}{|\vec{r}-\vec{r}\,'|} \; e\,\delta(\vec{r}\,'-\vec{r}(t'))$$

$$= \int dt' \; \frac{\delta\left(\frac{1}{c}|\vec{r}-\vec{r}(t')| - (t-t')\right)}{|\vec{r}-\vec{r}(t')|}$$

$$\equiv \int dt' \; \delta(g(t'))\, f(t') \qquad (2\cdot1)$$

where
$$\begin{cases} g(t') \equiv \frac{1}{c}|\vec{r}-\vec{r}(t')| - (t-t') \\ f_1(t') \equiv \dfrac{e}{|\vec{r}-\vec{r}(t')|} \end{cases}$$

By noticing the property of δ-function

$$\int \delta(g(x))\, f(x)\, dx = \frac{1}{|g'(x(0))|}\, f(x(0)) \qquad (2.2)$$

And if we let the root of $g(t')=0$ is $t'(0)=t'$

We can see,

$$g'(t') = \frac{\partial}{\partial t'} g(t')\Big|_{\text{at the root "}t'\text{"}}$$

$$= \frac{1}{c}\frac{\partial}{\partial t'}\sqrt{(\vec{r}-\vec{r}(t'))^2} + 1$$

$$= \frac{1}{c}\frac{(\vec{r}-\vec{r}(t'))\cdot\left(-\frac{d\vec{r}(t')}{dt'}\right)}{|\vec{r}-\vec{r}(t')|} + 1$$

$$= 1 - \frac{\vec{v}(t')\cdot[\vec{r}-\vec{r}(t')]}{c\,|\vec{r}-\vec{r}(t')|} \qquad \text{where } \vec{v}(t') = \frac{d}{dt}\vec{r}(t') \tag{2.3}$$

$$f(t') = \frac{e}{|\vec{r}-\vec{r}(t')|}$$

By (2.2) & (2.1)

$$\rightarrow \quad \phi(\vec{r},t) = \int dt'\,\delta(g(t'))\,f(t')$$

$$= \frac{1}{|g'(t'(0))|}\, f(t'(0)) \qquad \text{where root } t'(0) = t'$$

$$= \frac{1}{1-\frac{\vec{v}(t')\cdot[\vec{r}-\vec{r}(t')]}{c\,|\vec{r}-\vec{r}(t')|}} \cdot \frac{e}{|\vec{r}-\vec{r}(t')|}$$

$$= \frac{e}{|r-\vec{r}(t')| - (r-\vec{r}(t'))\cdot\frac{v(t')}{c}}$$

② From the retarded potential Eq (31.50)

$$\vec{A}(\vec{r},t) = \int (d\vec{r}')\, dt' \, \frac{\delta\left(\frac{1}{c}|\vec{r}-\vec{r}'| - (t-t')\right)}{|\vec{r}-\vec{r}'|} \, \frac{1}{c} \vec{j}(\vec{r}',t')$$

$$= \int dt' \, \frac{\delta\left(\frac{1}{c}|\vec{r}-\vec{r}(t')| - (t-t')\right) e\vec{v}(t')}{c\,|\vec{r}-\vec{r}(t')|}, \quad \text{where } \vec{j}(\vec{r},t) = e\vec{v}(t)\,\delta(\vec{r}-\vec{r}(t))$$

$$\equiv \int dt' \, \delta(g(t'))\, f_2(t') \qquad (2.4)$$

Now $$f_2(t') = \frac{e\vec{v}(t')}{c|\vec{r}-\vec{r}(t')|} = f_1(t')\frac{\vec{v}(t')}{c} \qquad (2.5)$$

By the similar method of part ①, (2.3) & (2.4) & (2.5) →

$$\vec{A}(\vec{r},t) = \frac{1}{|g'(t'(0))|} f_2(t'(0))$$

$$= \left[\frac{1}{|g'(t')|} f_1(t')\right] \frac{\vec{v}(t')}{c}$$

$$= \frac{\vec{v}(t')}{c}\phi(\vec{r},t)$$

where
$$\begin{cases} \phi(\vec{r},t) = \dfrac{f_1(t')}{|g'(t')|} \\ t'(0) = t'(g=0) = t', \text{ (root of } g(t')=0) \end{cases}$$

(b) From potentials of part (a).

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}$$

where

$$\begin{cases} \phi(\vec{r},t) = \dfrac{e}{|\vec{r}-\vec{r}_0(t_0')| - (\vec{r}-\vec{r}_0(t_0'))\dfrac{\vec{v}_0(t_0')}{c}} \\ \vec{A}(\vec{r},t) = \dfrac{\vec{v}_0(t_0')}{c}\,\phi(\vec{r},t) \end{cases}$$

(note: To avoid some confusion, we add sub-title "0")

Now we define

$$\begin{cases} \vec{R}^* \equiv \vec{r} - \vec{r}_0(t_0') \,, \quad \vec{v}^* = \vec{v}_0(t_0') \\ \vec{n} = \dfrac{\vec{R}^*}{R^*} \quad \text{direction of } \vec{R}^*, \text{ from source to field} \\ s^* \equiv |\vec{r}-\vec{r}_0(t_0')| - \dfrac{\vec{v}_0(t_0')}{c}\cdot(\vec{r}-\vec{r}_0(t_0')) \\ \quad = R^* - \dfrac{\vec{v}^*}{c}\cdot\vec{R}^* \end{cases}$$

By the definition of the retarded time

$$t - t_0' = |\vec{r}-\vec{r}_0(t_0')|/c$$

$$\rightarrow \quad t_0' - t + R^*/c = 0$$

$$\frac{\partial}{\partial t}: \quad \frac{\partial t_0'}{\partial t} - 1 + \frac{1}{c}\frac{\vec{R}^*}{R^*}\cdot(-\vec{v}^*)\frac{\partial t_0'}{\partial t} = 0$$

$$\left(1 - \frac{\vec{v}^*}{c}\cdot\frac{\vec{R}^*}{R^*}\right)\frac{\partial t_0'}{\partial t} = 1$$

$$\rightarrow \quad \frac{\partial t_0'}{\partial t} = R^*/s^*$$

Also $$\frac{\partial R^*}{\partial t_0'} = \frac{\vec{R}^*}{R^*} \cdot (-\vec{v}^*)$$

$$\nabla t_0' = -\frac{1}{c} \nabla R^*$$

$$= \frac{\vec{R}^*}{R^*} \cdot (\vec{\vec{I}} - \vec{v}^* \cdot \nabla t_0')$$

$$= \frac{\vec{R}^* - (\vec{R}^* \cdot \vec{v}^*)\, \nabla t_0'}{R^*}$$

$$\rightarrow \quad \nabla t_0' = -\frac{1}{c}\left[\frac{\vec{R}^*}{R^*} - \frac{1}{R^*}(\vec{R}^* \cdot \vec{v}^*)\nabla t_0'\right]$$

$$= -\frac{1}{c}\,\frac{\vec{R}^*/R^*}{1 - \frac{1}{cR^*}\vec{R}^* \cdot \vec{v}^*}$$

$$= -\frac{\vec{R}^*}{cs^*} \tag{2.7}$$

Also $$\nabla R^* = \vec{R}^*/s^* \tag{2.8}$$

$$\frac{\partial}{\partial t} R^* = \frac{\partial R^*}{\partial t_0'}\,\frac{\partial t_0'}{\partial t} = -\frac{\vec{v}^* \cdot \vec{R}^*}{s^*} \tag{2.9}$$

Hence, by using (2.6) ~ (2.9), we can obtain

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \quad \text{where } \phi = e/s^*$$

$$= \frac{e}{s^{*2}} \nabla s^* - \left[\frac{1}{c^2}\frac{\partial \vec{v}^*}{\partial t}\phi + \frac{\vec{v}^*}{c}\frac{\partial \phi}{\partial t}\right]$$

$$= \frac{e}{s^{*2}}\left[\nabla s^* + \frac{1}{c^2}\vec{v}^*\frac{\partial s^*}{\partial t} - \frac{1}{c^2}\vec{a}^* R^*\right] \tag{2.10}$$

Now we need to calculate ∇S^* & $\partial S^*/\partial t$

$$\nabla S^* = \nabla R^* - (\nabla \vec{R}^*)\cdot \vec{v}^*/c - (\vec{R}^*\cdot \vec{a}^*/c)\,\nabla t_0'$$

$$= \vec{R}^*/S^* - (\overleftrightarrow{I} - \vec{v}^*\nabla t_0')\cdot\frac{\vec{v}^*}{c} - \left(\vec{R}^*\cdot\frac{\vec{a}^*}{c}\right)\left(-\frac{\vec{R}^*}{cS^*}\right)$$

$$= \frac{\vec{R}^*}{S^*} - \frac{\vec{v}^*}{c} - \frac{v^{*2}}{c}\left(\frac{\vec{R}^*}{cS^*}\right) + (\vec{R}^*\cdot\vec{a}^*)\frac{\vec{R}^*}{c^2S^*}$$

$$= \left(1-\frac{v^{*2}}{c^2}\right)\frac{\vec{R}^*}{S^*} - \frac{\vec{v}^*}{c} + \frac{(\vec{R}^*\cdot\vec{a}^*)}{c^2S^*}\,\vec{R}^* \qquad (2.11)$$

$$\frac{\partial S^*}{\partial t} = \frac{\partial R^*}{\partial t} - \frac{\partial \vec{R}^*}{\partial t}\cdot\frac{\vec{v}^*}{c} - \vec{R}^*\cdot\frac{\vec{a}^*}{c}\frac{\partial t_0'}{\partial t}$$

$$= -\frac{\vec{v}^*\cdot\vec{R}^*}{S^*} + \frac{v^{*2}R^*}{cS^*} - \frac{R^*}{cS^*}(\vec{R}^*\cdot\vec{a}^*) \qquad (2.12)$$

(2.11) & (2.12) set into (2.10)

$$\rightarrow \vec{E} = \frac{e}{S^{*3}}\Big[\left(1-\frac{v^{*2}}{c^2}\right)\vec{R}^* - \frac{\vec{v}^*}{c}S^* + \frac{\vec{R}^*\cdot\vec{a}^*}{c^2}\vec{R}^*$$
$$+ \frac{\vec{v}^*}{c^2}\left(-\vec{v}^*\cdot\vec{R}^* + \frac{v^{*2}R^*}{c} - \frac{R^*}{c}(\vec{R}^*\cdot\vec{a}^*)\right) - \frac{S^*R^*}{c^2}\vec{a}^*\Big]$$

$$= \frac{e}{S^{*3}}\left\{\left(1-\frac{v^{*2}}{c^2}\right)\left(\vec{R}^* - \frac{R^*}{c}\vec{v}^*\right) + \frac{1}{c^2}\vec{R}^*\times\left[\left(\vec{R}^* - \frac{R^*}{c}\vec{v}^*\right)\times\vec{a}^*\right]\right\}$$

$$= e\left(\frac{\vec{n}-\vec{\beta}}{\gamma^2(1-\vec{\beta}\cdot\vec{n})^3R^{*2}}\right) + \frac{e}{c}\,\frac{\vec{n}\times[(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\vec{n})^3R^*} \qquad (2.13)$$

↓ rest decay fast

↓ Radiation significant

To calculate $\vec{B}$, a good way is to relate it to $\vec{E}$

$$\vec{B} = \nabla \times \vec{A}$$

$$= \nabla \times \left(\frac{\vec{v}_0(t')}{c}\, \phi(\vec{r}, t) \right)$$

$$= \nabla\phi \times \frac{\vec{v}^*}{c} + \phi \nabla \times \frac{\vec{v}^*}{c}$$

$$= \nabla\phi \times \frac{\vec{v}^*}{c} + \frac{\phi \nabla t'}{c} \times \frac{\partial \vec{v}^*}{\partial t'}$$

$$= -\frac{e}{S^{*2}} \nabla S^* \times \frac{\vec{v}^*}{c} + \frac{\phi}{c}\left(-\frac{\vec{R}^*}{cR^*} \frac{\partial t'}{\partial t} \right) \times \frac{\partial \vec{v}^*}{\partial t'}$$

$$= -\frac{e^2}{S^{*2}} \left(\frac{\vec{R}^*}{R^*} - \frac{\vec{v}^*}{c} + \frac{\partial S^*}{\partial t'} \nabla t' \right) \times \frac{\vec{v}^*}{c} + \frac{\vec{R}^*}{R^*} \times \left(-\frac{\phi}{c^2}\, \vec{v}^* \frac{\partial t'}{\partial t} \right)$$

$$= \frac{\vec{R}^*}{R^*} \times \left(-\frac{e}{S^{*2}} \frac{\vec{v}^*}{c} + \frac{\vec{v}^*}{c^2} \frac{e}{S^{*2}} \frac{\partial S^*}{\partial t} \right) + \frac{\vec{R}^*}{R^*} \times \left(-\frac{\phi}{c^2} \frac{\partial \vec{v}^*}{\partial t} \right)$$

$$= \vec{n} \times \left(\frac{e}{S^{*2}} \nabla S^* - \frac{\vec{v}^*}{c^2} \frac{\partial \phi}{\partial t} - \frac{\phi}{c^2} \frac{\partial \vec{v}^*}{\partial t} \right)$$

$$= \vec{n} \times \left(-\nabla\phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right)$$

(2.13) $\rightarrow$ $$= \vec{n} \times \vec{E}$$

$$= \vec{n} \times \left(e \frac{\vec{n} - \vec{\beta}}{\gamma^2 (1 - \vec{\beta}\cdot\vec{n})^3 R^{*2}} + \frac{e}{c} \frac{\vec{n} \times [(\vec{n} - \vec{\beta}) \times \dot{\vec{\beta}}]}{(1 - \vec{\beta}\cdot\vec{n})^3 R^*} \right)$$

$$= -e \frac{\vec{n} \times \vec{\beta}}{\gamma^2 (1 - \vec{\beta}\cdot\vec{n})^3 R^{*2}} + \frac{e}{c} \frac{\vec{n} \times [\vec{n} \times [(\vec{n} - \vec{\beta}) \times \dot{\vec{\beta}}]]}{(1 - \vec{\beta}\cdot\vec{n})^3 R^*} \qquad (2.14)$$

$\downarrow$ rest decay fast

$\downarrow$ radiation significant

3. Radiation of an electron moving in a magnetic field

For a nonrelativistic particle

$$(2.13) \rightarrow \quad \vec{E} \simeq \frac{e\vec{n}}{\gamma^2 R^{*2}} + \frac{e}{c}\frac{\vec{n}\times(\vec{n}\times\dot{\vec{\beta}})}{R^*}$$

Coulomb law, decay fast

$$\simeq \frac{e}{c}\frac{\vec{n}\times(\vec{n}\times\dot{\vec{\beta}})}{R^*}$$

$$\vec{B} = \vec{n}\times\vec{E}$$

$$\simeq \frac{e}{c}\frac{\vec{n}\times[\vec{n}\times(\vec{n}\times\dot{\vec{\beta}})]}{R^*}$$

$$= -\frac{e}{c}\frac{\vec{n}\times\dot{\vec{\beta}}}{R^*}$$

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{c}{4\pi}|\vec{E}_{rad}|^2\,\vec{n}$$

$$= \frac{c}{4\pi}E^2\,\vec{n}$$

$$\frac{dP}{d\Omega} = \frac{c}{4\pi}|\vec{R}^*\cdot\vec{E}_{rad}|^2 = \frac{e^2}{4\pi c}|\vec{n}\times(\vec{n}\times\dot{\vec{\beta}})|^2 = \frac{e^2}{4\pi c^3}|\dot{\vec{v}}^*|^2\sin^2\theta.$$

$$\rightarrow P = \int \frac{dP}{d\Omega}\,d\Omega$$

$$= \int \frac{e^2}{4\pi c^3}|\dot{\vec{v}}^*|^2\sin^2\theta\,d\Omega$$

$$= \frac{2e^2}{3c^3}|\dot{\vec{v}}|^2 \quad \leftarrow \text{where } m\dot{\vec{v}} = \frac{e}{c}\vec{v}\times\vec{B}$$

$$= \frac{2e^2}{3c^3}\left(\frac{evB}{mc}\right)^2 = \frac{4e^4B^2}{3m^3c^5}E, \quad \text{where } E = \tfrac{1}{2}mv^2$$

Hence, we have calculated the radiated power

$$P = \frac{4e^4 B^2}{3m^3 c^5} E$$

$$\equiv \gamma E \qquad (3.1)$$

Also $-P = dE/dt$ (3.2)

(3.1) & (3.2) $\rightarrow \gamma E = P = -\frac{dE}{dt}$

$\rightarrow -\frac{dE}{dt} = \gamma E$

And $\gamma = \frac{4e^4 B^2}{3m^3 c^5}$

For a electron $m = m_e = 9.1 \times 10^{-28}$ g

$$\frac{1}{\gamma} = \frac{3m^3c^5}{4e^4B^2}$$

$$= \frac{3 \times (9.1 \times 10^{-28}\ g)^3 (3 \times 10^{10}\ cm/s)^5}{4 \times (4.8 \times 10^{-10}\ esu)^4 (10^4\ Gauss)^2}$$

$$= 2.59\ [Sec]$$

4. Radiation of a classical hydrogen atom.

From prob. 3, we have known

$$P = \frac{2e^2}{3c^3}|\dot{\vec{v}}|^2 \qquad \leftarrow \qquad m\dot{\vec{v}} = -\frac{e^2\vec{r}}{r^3}$$

$$= \frac{2e^2}{3c^3}\left|\frac{e^2}{mr^2}\right|^2$$

$$= \frac{2e^6}{3m^2c^3r^4} \qquad (4.1)$$

$V(r) = -\frac{e^2}{r}$, Virial Theory $\rightarrow$ $E = -\frac{e^2}{2r}$

$\therefore$ (4.1) $\rightarrow$ $P = \frac{2e^6}{3m^2c^3r^4}$

$$= \frac{2^5}{3m^2c^3e^2}\left(-\frac{e^2}{2r}\right)^4$$

$$= \frac{32}{3m^2c^3e^2}E^4 \qquad (4.2)$$

By $-P = \frac{dE}{dt}$ $\rightarrow$ $-\frac{dE}{dt} = P = \frac{32}{3m^2c^3e^2}E^4$ (4.3)

Non-linear ↑

To integrate, (4.3) $\rightarrow$

$$-\frac{dE}{E^4} = \frac{32}{3m^2c^3e^2}dt$$

The time for E_2 (initial) to E_1 (final)

$$T = t_1 - t_2 = \frac{m^2c^3e^2}{32}\left(\frac{1}{E_1^3} - \frac{1}{E_2^3}\right)$$

$$= \frac{m^2c^3e^2}{32}\frac{E_2^3 - E_1^3}{(E_1E_2)^3} \qquad (4.4)$$

The electron reaches the center $r_2 \to r_1 = 0$

$$\underset{\uparrow \text{initial}}{E_2} \to \underset{\uparrow \text{final}}{E_1} = -\frac{e^2}{2r_1} = -\infty$$

By (4.4)

$$T_f = \frac{m^2c^3e^2}{32}\left(-0 - \frac{1}{E_2^3}\right)$$

$$= -\frac{m^2c^3e^2}{32\,E_2^3}$$

So the electron reaches the center in a finite time T_f

If an electron starts from $r_0 = 10^{-8}$ cm $= r_2$ (initial radius)

$\rightarrow$ $E_2 = -\frac{e^2}{2r_2} = -\frac{e^2}{2r_0}$ (initial state)

It takes the time

$$T = -\frac{m^2c^3e^2}{32\left(-\frac{e^2}{2r_0}\right)^3}$$

$$= \frac{m^2c^3r_0^3}{4e^4}$$

$$= \frac{(9.1\times10^{-28}\,g)^2\,(3\times10^{10}\,cm/s)^3\,(10^{-8}\,cm)^3}{4\times(4.8\times10^{-10}\,esu)^4}$$

$$= 1.053\times10^{-10}\ s$$

$\simeq 0.1$ ns

5. Dipole radiation.

(a). For a non-relativistic electron (m, e) in an oscillating electric field $\vec{E} = E_0 \hat{z} \cos\omega t$

The Eq of motion should be.

$$\vec{F} = m\ddot{\vec{r}} = e\vec{E} = eE_0 \hat{z} \cos\omega t \qquad (5.1)$$

If we choose the initial position of the electron at the origin $\vec{r}(t=0) = 0$

and assume the initial velocity $\dot{\vec{r}}(t=0) = 0$

then (5.1) can be solved as.

$$\dot{\vec{r}}(t) = \frac{eE_0}{m\omega} \sin\omega t \ \hat{z}$$

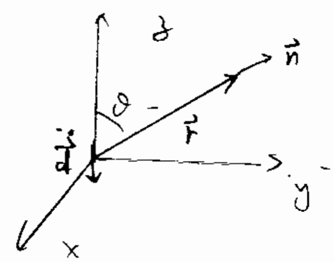
$$\rightarrow \quad \vec{r}(t) = -\frac{eE_0}{m\omega^2} \cos\omega t \ \hat{z}$$

Hence, the oscillating electric field actually excite the electron to be like an oscillating dipole

$$\vec{d}(t) = e\vec{r}(t) = -\frac{e^2 E_0}{m\omega^2} \cos\omega t \ \hat{z} \qquad (5.2)$$

with an oscillating frequency ω.

Now, we consider the radiation power



$$\because \quad \vec{S}_{rad} = \frac{1}{4\pi c^3 r^2} (\ddot{\vec{d}} \times \vec{n})^2 \vec{n}$$

(Note: Here $\vec{r}$ is the distance between the field and the source (dipole)
do not confuse with the one in the dipole expression of (5.2))

The outward component of $\vec{S}_{rad}$ is

$$\vec{S}_{rad} \cdot \vec{n} = \frac{\ddot{\vec{d}}^2}{4\pi c^3 r^2} \sin^2\theta$$

So the power emitted per unit solid angle is

$$\frac{dP}{d\Omega} = (\vec{S}_{rad} \cdot \vec{n}) \, r^2$$

$$= \frac{\ddot{\vec{d}}^2}{4\pi c^3} \sin^2\theta$$

But the meaningful quantity is the radiated power averaged over one complete period of oscillation of the system.

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\langle \ddot{\vec{d}} \rangle^2}{4\pi c^3} \sin^2\theta \qquad (5.3)$$

By Eq (5.2) $\langle \ddot{\vec{d}}^2 \rangle = \frac{1}{m^2} e^4 E_0^2 \langle \cos^2 \omega t \rangle = \frac{1}{2m^2} e^4 E_0^2$

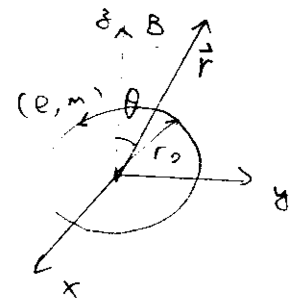
Set it into (5.3), we can eventually obtain

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^4 E_0^2}{8\pi c^3 m^2} \sin^2\theta$$

(b). A nonrelativistic electron of (m, e) should have rotating motion in a uniform & constant $\vec{B} \equiv B\hat{z}$

And the eq of motion

$$\frac{e\dot{\vec{r}}}{c} \times \vec{B} = \vec{F} = m\ddot{\vec{r}} \qquad (5.4)$$



The electron on the dipole rotate in $x-y$ plane

By $\vec{d}(t) = e\vec{r}(t)$

We can rewrite (5.4) as

$$\ddot{\vec{d}} = \frac{e}{m}\dot{\vec{d}} \times \vec{B}$$

So $\vec{d}$ is actually a rotating dipole with frequency $\vec{\omega} = e\vec{B}/m$

By choosing a suitable coordinate frame we should have $\vec{d} = e\vec{r}(t) = er_0(\cos\omega t\,\hat{e}_x + \sin\omega t\,\hat{e}_y)$, where $\omega = \frac{eB}{m}$

Hence, $\vec{S}_{rad} \cdot \vec{n} = \frac{\ddot{\vec{d}}^{\,2}}{4\pi c^3 r^2}\sin^2\theta$ as in part (a).

[Note : $\vec{r}$ and $\vec{r}(t)$ are seperatly (the distance between the field & the source) and (the value for dipole), don't confuse]

$$\langle \sin^2\theta \rangle = \langle(-\sin\omega t\,\hat{e}_x + \cos\omega t\,\hat{e}_y)(\sin\xi\cos\varphi\,\hat{e}_x + \sin\xi\sin\varphi\,e_y + \cos\xi\,e_z)\rangle$$

$$= \frac{1+\cos^2\xi}{2}$$

$$\rightarrow \left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{8\pi c^5 m^2}(e^2 VB)^2(1+\cos^2\xi)$$