Lecture 18 (Nov. 13, 2017)

18.1   Symmetries in Quantum Mechanics

A symmetry is a physical operation we can perform on the system that leaves the physics unchanged. As an example, consider a free particle,

$$H_{\text{free}} = \frac{p^2}{2m} \, . \tag{18.1}$$

This Hamiltonian does not depend on position, so we can translate $x \to x + a$. Under this transformation, H_{free} is unchanged and $[x + a, p] = i\hbar$, implying that both H and the commutation algebra are unchanged. Thus, translation is a symmetry of the free-particle system. This Hamiltonian has several other symmetries. *Inversion* (or *parity*), given by $x \to -x, p \to -p$, preserves $[x, p] = i\hbar$ and H_{free}, and so is a symmetry. *Time reversal*, which sends $t \to -t$, is also a symmetry; we will discuss its realization in quantum mechanics later. The system is also invariant under Galilean transformations, even though Galilean boosts $p \to p + mv_0$ seem to change the Hamiltonian. Galilean boosts change the phase in the path integral by something that does not depend on the path, so the probabilities do not change, but the amplitudes do; this example is more subtle, and so we will stick to discussions of the other symmetries of the system.

18.1.1   Symmetry Transformations

We define a *symmetry operation* as a linear transformation $U \colon |a\rangle \to |a'\rangle$ such that all results of measurement are preserved. This means that

$$\left| \langle b' | a' \rangle \right|^2 = |\langle b | a \rangle|^2 \tag{18.2}$$

for all $a, b \in \mathcal{H}$. From the definition of the primed kets, this gives

$$\left| \langle b | U^\dagger U | a \rangle \right|^2 = |\langle b | a \rangle|^2 \, . \tag{18.3}$$

Wigner's theorem tells us that if U satisfies this condition, then U must be unitary or anti-unitary; an *anti-unitary* operator $\tilde{U}$ is one that satisfies

$$\tilde{U}|a\rangle = KU|U\rangle = (U|a\rangle)^\dagger \, , \tag{18.4}$$

where K is the complex conjugation operator, and U is some unitary operator. If we assume that U is linear or *anti-linear*, meaning that

$$U\left(\sum_a c_a |a\rangle \right) = \sum_a c_a^* U |a\rangle \, , \tag{18.5}$$

then we can show Wigner's theorem fairly simply (this will appear on your homework). Wigner's theorem can be proved without making this assumption, but the proof is more subtle.

18.1.2   Continuous Symmetries and Conservation Laws

Consider a symmetry operation described by a unitary operator U. This is a symmetry of the Hamiltonian if H is unchanged by the action of U, i.e.

$$H = U^\dagger H U \, , \tag{18.6}$$

i.e., $[H, U] = 0$. If the symmetry transformation can be continuously built up as a series of infinitesimal transformations starting from the identity operator $\mathbb{1}$, then we call it a *continuous symmetry*. Translation and rotation are examples of continuous symmetries: any translation or rotation can be built up from the identity by doing a sequence of infinitesimal translations or rotations. Symmetries that cannot be built up in this way are called *discrete symmetries*. Parity is an example of a discrete symmetry (in even dimensions, we can construct parity as a rotation, but it is nonetheless possible that the Hamiltonian is symmetric under parity but not rotations, so parity is still discrete).

Let us now consider continuous symmetries. All of the information about a continuous symmetry is contained in its infinitesimal form, because we can always build up a continuous transformation from its infinitesimal version. As we have seen previously, an infinitesimal unitary operator can be expressed in terms of a Hermitian operator: if we expand an infinitesimal unitary operator U as

$$U = 1 - \frac{i\epsilon}{\hbar}G + O(\epsilon^2),\tag{18.7}$$

then the statement $U^\dagger U = \mathbb{1}$ implies $G^\dagger = G$, i.e., G is Hermitian. The statement that U is a symmetry is $U^\dagger H U = H$, i.e.,

$$\left(1 + \frac{i\epsilon}{\hbar}G + O(\epsilon^2)\right)H\left(1 - \frac{i\epsilon}{\hbar}G + O(\epsilon^2)\right) = H,\tag{18.8}$$

which gives us $[G, H] = 0$. Because G is Hermitian, it corresponds to some observable, and we now have

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \frac{1}{i\hbar}[G, H] = 0,\tag{18.9}$$

so we see that G is conserved. This leads us to the following general statement:

Theorem 4 (Noether's Theorem). *For every continuous symmetry of the Hamiltonian in quantum mechanics, there is a corresponding conserved quantity. Conversely, if some observable G is conserved, then $[G, H] = 0$, and we can define unitary operators*

$$U(\theta) = e^{i\theta G/\hbar},\tag{18.10}$$

which will satisfy $U(\theta)^\dagger H U(\theta) = H$, showing that $U(\theta)$ is a continuous symmetry of the Hamiltonian.

18.1.3   Translations

As an example, consider again translation, $x \to x + a$. The translation operator is

$$T(a) = e^{-ipa/\hbar}.\tag{18.11}$$

This satisfies $(T(a))^\dagger = (T(a))^{-1}$, i.e., the translation operator is unitary. It has the following properties:

- $(T(a))^{-1} = T(-a)$,

- $T(a')T(a'') = T(a' + a'')$,

- $T^\dagger(a)xT(a) = x + a$.

For infinitesimal a, we can expand the translation operator as

$$T(a) \approx 1 - \frac{ia}{\hbar} p \, , \qquad (18.12)$$

where p is the Hermitian *generator* of $T(a)$. If a system has translation symmetry, then momentum is conserved, from Noether's theorem. In d dimensions, we generalize to translations $x_i \to x_i + a_i$, with $i = 1, \ldots, d$. Then, the unitary translation operator is

$$T(\{a_i\}) = \prod_i T_i(a_i) = \prod_i e^{-i p_i a_i / \hbar} \, . \qquad (18.13)$$

It is a geometric fact that translations in different directions commute, $[T_i(a_i), T_j(a_j)] = 0$, for all i, j. This tells us that the Hermitian generators commute, $[p_i, p_j] = 0$, for all i, j. We can define momentum as the observable that corresponds to the Hermitian generator of translations; then taking a to be infinitesimal in the identity

$$e^{ipa/\hbar} x e^{-ipa/\hbar} = x + a \qquad (18.14)$$

gives us the commutation relation $[x, p] = i\hbar$.

18.1.4 Time Translations

For a closed system, time translation is a symmetry, and the corresponding unitary operator is the time-evolution operator,

$$U(t) = e^{-iHt/\hbar} \, . \qquad (18.15)$$

The Hamiltonian H is the generator of this symmetry, and so the fact that the system is invariant under time translation implies the conservation of energy,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0 \, . \qquad (18.16)$$

18.1.5 Rotations

If we have a d-dimensional system with position coordinate x_i, then a rotation is given by

$$x_i \to x_i' = \sum_{j=1}^{d} R_{ij} x_j = R_{ij} x_j \, , \qquad (18.17)$$

such that the scalar product is preserved,

$$\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}' \cdot \boldsymbol{y}' \, . \qquad (18.18)$$

Here we are using the Einstein summation convention that repeated indices are summed over. The preservation of the scalar product can be written as

$$R_{ij} R_{ik} x_j y_k = x_i y_i \, , \qquad (18.19)$$

which implies

$$R_{ij} R_{ik} = \delta_{jk} \, , \qquad (18.20)$$

i.e., $RR^{\mathrm{T}} = \mathbb{1}$. That is, R is an orthogonal matrix.

The orthogonality condition, $RR^{\mathrm{T}} = \mathbb{1}$, implies that $(\det R)^2 = 1$, meaning that $\det R = \pm 1$. We see then that there are two classes of rotations. Matrices with $\det R = -1$ are not continuously connected to the identity; they involve inversions $x_i \to -x_i$ combined with an ordinary (orientation-preserving) rotation. (Note that in even dimensions, parity is orientation-preserving.) For now, we will stick to considering rotations R that have $\det R = +1$. Each rotation R is represented in quantum mechanics by a unitary operator $\mathcal{D}(R)$ that acts on the system's Hilbert space.

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