Electromechanical Dynamics

Appendix E

SUMMARY OF PARTS I AND II AND USEFUL THEOREMS

IDENTITIES

$$\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{B} \times \mathbf{C},$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi,$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B},$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B},$$

$$\nabla(\phi\psi) = \phi\, \nabla\psi + \psi\, \nabla\phi,$$

$$\nabla \cdot (\psi\mathbf{A}) = \mathbf{A} \cdot \nabla\psi + \psi\nabla \cdot \mathbf{A},$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B},$$

$$\nabla \cdot \nabla\phi = \nabla^2\phi,$$

$$\nabla \cdot \nabla \times \mathbf{A} = 0,$$

$$\nabla \times \nabla\phi = 0,$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A},$$

$$(\nabla \times \mathbf{A}) \times \mathbf{A} = (\mathbf{A} \cdot \nabla)\mathbf{A} - \tfrac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A}),$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$$

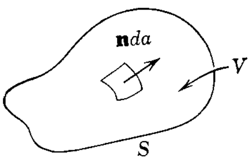
$$\nabla \times (\phi\mathbf{A}) = \nabla\phi \times \mathbf{A} + \phi\nabla \times \mathbf{A},$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}.$$

THEOREMS

$$\int_a^b \nabla\phi \cdot d\mathbf{l} = \phi_b - \phi_a.$$

Divergence theorem $\quad \oint_S \mathbf{A} \cdot \mathbf{n}\, da = \int_V \nabla \cdot \mathbf{A}\, dV$



Stokes's theorem $\quad \oint_C \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, da$

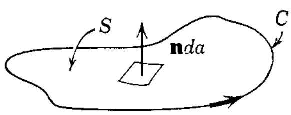
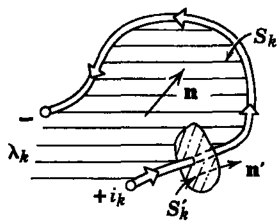
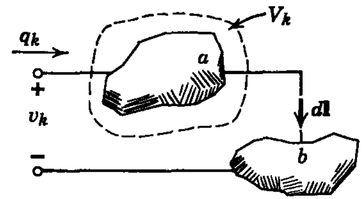


Table 1.2 Summary of Quasi-Static Electromagnetic Equations

	Differential Equations		Integral Equations	
Magnetic field system	$\nabla \times \mathbf{H} = \mathbf{J}_f$	(1.1.1)	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot \mathbf{n}\, da$	(1.1.20)
	$\nabla \cdot \mathbf{B} = 0$	(1.1.2)	$\oint_S \mathbf{B} \cdot \mathbf{n}\, da = 0$	(1.1.21)
	$\nabla \cdot \mathbf{J}_f = 0$	(1.1.3)	$\oint_S \mathbf{J}_f \cdot \mathbf{n}\, da = 0$	(1.1.22)
	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	(1.1.5)	$\oint_C \mathbf{E}' \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot \mathbf{n}\, da$ where $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$	(1.1.23)
Electric field system	$\nabla \times \mathbf{E} = 0$	(1.1.11)	$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	(1.1.24)
	$\nabla \cdot \mathbf{D} = \rho_f$	(1.1.12)	$\oint_S \mathbf{D} \cdot \mathbf{n}\, da = \int_V \rho_f\, dV$	(1.1.25)
	$\nabla \cdot \mathbf{J}_f = -\frac{\partial \rho_f}{\partial t}$	(1.1.14)	$\oint_S \mathbf{J}_f' \cdot \mathbf{n}\, da = -\frac{d}{dt}\int_V \rho_f\, dV$	(1.1.26)
	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$	(1.1.15)	$\oint_C \mathbf{H}' \cdot d\mathbf{l} = \int_S \mathbf{J}_f' \cdot \mathbf{n}\, da + \frac{d}{dt}\int_S \mathbf{D} \cdot \mathbf{n}\, da$ where $\mathbf{J}_f' = \mathbf{J}_f - \rho_f \mathbf{v}$ $\mathbf{H}' = \mathbf{H} - \mathbf{v} \times \mathbf{D}$	(1.1.27)

Table 2.1 Summary of Terminal Variables and Terminal Relations

Magnetic field system	Electric field system
	

Definition of Terminal Variables

Flux	Charge
$\lambda_k = \int_{S_k} \mathbf{B} \cdot \mathbf{n}\, da$	$q_k = \int_{V_k} \rho_f\, dV$

Current	Voltage
$i_k = \int_{S_k'} \mathbf{J}_f \cdot \mathbf{n}'\, da$	$v_k = \int_a^b \mathbf{E} \cdot d\mathbf{l}$

Terminal Conditions

$v_k = \dfrac{d\lambda_k}{dt}$	$i_k = \dfrac{dq_k}{dt}$
$\lambda_k = \lambda_k(i_1 \cdots i_N;\ \text{geometry})$	$q_k = q_k(v_1 \cdots v_N;\ \text{geometry})$
$i_k = i_k(\lambda_1 \cdots \lambda_N;\ \text{geometry})$	$v_k = v_k(q_1 \cdots q_N;\ \text{geometry})$

Table 3.1 Energy Relations for an Electromechanical Coupling Network with N Electrical and M Mechanical Terminal Pairs*

Magnetic Field Systems		Electric Field Systems	
Conservation of Energy			
$dW_m = \sum_{j=1}^{N} i_j \, d\lambda_j - \sum_{j=1}^{M} f_j^e \, dx_j$	(a)	$dW_e = \sum_{j=1}^{N} v_j \, dq_j - \sum_{j=1}^{M} f_j^e \, dx_j$	(b)
$dW_m' = \sum_{j=1}^{N} \lambda_j \, di_j + \sum_{j=1}^{M} f_j^e \, dx_j$	(c)	$dW_e' = \sum_{j=1}^{N} q_j \, dv_j + \sum_{j=1}^{M} f_j^e \, dx_j$	(d)
Forces of Electric Origin, $j = 1, \ldots, M$			
$f_j^e = -\dfrac{\partial W_m(\lambda_1, \ldots, \lambda_N; x_1, \ldots, x_M)}{\partial x_j}$	(e)	$f_j^e = -\dfrac{\partial W_e(q_1, \ldots, q_N; x_1, \ldots, x_M)}{\partial x_j}$	(f)
$f_j^e = \dfrac{\partial W_m'(i_1, \ldots, i_N; x_1, \ldots, x_M)}{\partial x_j}$	(g)	$f_j^e = \dfrac{\partial W_e'(v_1, \ldots, v_N; x_1, \ldots, x_M)}{\partial x_j}$	(h)
Relation of Energy to Coenergy			
$W_m + W_m' = \sum_{j=1}^{N} \lambda_j i_j$	(i)	$W_e + W_e' = \sum_{j=1}^{N} v_j q_j$	(j)
Energy and Coenergy from Electrical Terminal Relations			
$W_m = \sum_{j=1}^{N} \int_0^{\lambda_j} i_j(\lambda_1, \ldots, \lambda_{j-1}, \lambda_j', 0, \ldots, 0; x_1, \ldots, x_M) \, d\lambda_j'$	(k)	$W_e = \sum_{j=1}^{N} \int_0^{q_j} v_j(q_1, \ldots, q_{j-1}, q_j', 0, \ldots, 0; x_1, \ldots, x_M) \, dq_j'$	(l)
$W_m' = \sum_{j=1}^{N} \int_0^{i_j} \lambda_j(i_1, \ldots, i_{j-1}, i_j', 0, \ldots, 0; x_1, \ldots, x_M) \, di_j'$	(m)	$W_e' = \sum_{j=1}^{N} \int_0^{v_j} q_j(v_1, \ldots, v_{j-1}, v_j', 0, \ldots, 0; x_1, \ldots, x_M) \, dv_j'$	(n)

* The mechanical variables f_j and x_j can be regarded as the jth force and displacement or the jth torque T_j and angular displacement θ_j.

Table 6.1 Differential Equations, Transformations, and Boundary Conditions for Quasi-static Electromagnetic Systems with Moving Media

	Differential Equations		Transformations		Boundary Conditions	
Magnetic field systems	$\nabla \times \mathbf{H} = \mathbf{J}_f$	(1.1.1)	$\mathbf{H}' = \mathbf{H}$	(6.1.35)	$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K}_f$	(6.2.14)
	$\nabla \cdot \mathbf{B} = 0$	(1.1.2)	$\mathbf{B}' = \mathbf{B}$	(6.1.37)	$\mathbf{n} \cdot (\mathbf{B}^a - \mathbf{B}^b) = 0$	(6.2.7)
	$\nabla \cdot \mathbf{J}_f = 0$	(1.1.3)	$\mathbf{J}_f' = \mathbf{J}_f$	(6.1.36)	$\mathbf{n} \cdot (\mathbf{J}_f{}^a - \mathbf{J}_f{}^b) + \nabla_\Sigma \cdot \mathbf{K}_f = 0$	(6.2.9)
	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	(1.1.5)	$\mathbf{E}' = \mathbf{E} + \mathbf{v}^r \times \mathbf{B}$	(6.1.38)	$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = v_n(\mathbf{B}^a - \mathbf{B}^b)$	(6.2.22)
	$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$	(1.1.4)	$\mathbf{M}' = \mathbf{M}$	(6.1.39)		
Electric field systems	$\nabla \times \mathbf{E} = 0$	(1.1.11)	$\mathbf{E}' = \mathbf{E}$	(6.1.54)	$\mathbf{n} \times (\mathbf{E}^a - \mathbf{E}^b) = 0$	(6.2.31)
	$\nabla \cdot \mathbf{D} = \rho_f$	(1.1.12)	$\mathbf{D}' = \mathbf{D}$	(6.1.55)	$\mathbf{n} \cdot (\mathbf{D}^a - \mathbf{D}^b) = \sigma_f$	(6.2.33)
			$\rho_f' = \rho_f$	(6.1.56)		
	$\nabla \cdot \mathbf{J}_f = -\dfrac{\partial \rho_f}{\partial t}$	(1.1.14)	$\mathbf{J}_f' = \mathbf{J}_f - \rho_f \mathbf{v}^r$	(6.1.58)	$\mathbf{n} \cdot (\mathbf{J}_f{}^a - \mathbf{J}_f{}^b) + \nabla_\Sigma \cdot \mathbf{K}_f = v_n(\rho_f{}^a - \rho_f{}^b) - \dfrac{\partial \sigma_f}{\partial t}$	(6.2.36)
	$\nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t}$	(1.1.15)	$\mathbf{H}' = \mathbf{H} - \mathbf{v}^r \times \mathbf{D}$	(6.1.57)	$\mathbf{n} \times (\mathbf{H}^a - \mathbf{H}^b) = \mathbf{K}_f + v_n \mathbf{n} \times [\mathbf{n} \times (\mathbf{D}^a - \mathbf{D}^b)]$	(6.2.38)
	$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$	(1.1.13)	$\mathbf{P}' = \mathbf{P}$	(6.1.59)		